Exercises. These do not need to be turned in.

1. Compute the Fourier transforms of the following:
   (a) \( f(x_1, \cdots, x_d) = e^{-|x_1| + \cdots + |x_d|} \)
   (b) \( f(x) = \cos(x) g(x) \); express your answer in terms of \( \hat{g} \)
   (c) \( f(x) = \sin(x) g(x) \); express your answer in terms of \( \hat{g} \)
   (d) \( f(x, y) = \cos(x) \sin(y) g(x, y) \); express your answer in terms of \( \hat{g} \) (note correction)
   (e) \( f(x, y, z) = \cos(x) \sin(y) e^{-z^2} \)
   (f) \( f(x, y) = \frac{\partial^2}{\partial x \partial y} \left[ \cos(x + y) e^{-x^2 - y^2}/2 \right] \)
   (g) \( f(x, y, z) = \begin{cases} 1, & (x, y, z) \in [-1, 1]^3 \\ 0, & \text{otherwise} \end{cases} \)

2. For each of the following PDEs, find the differential equation (in \( t \)) satisfied by the spatial Fourier transform \( \hat{u}(\omega, t) \) of \( u \). Note that you do not need to solve the differential equation.
   (a) \( u_t = (A + B \cos(\lambda x)) u_{xx} \)
      with \( u = u(x, t) \), \((x, t) \in \mathbb{R} \times [0, \infty)\), and \( A, B, \lambda > 0 \)
   (b) \( u_{tt} = (A + B \cos(\lambda x) \sin^2(\eta y)) \Delta u \)
      with \( u = u(x, y, t) \), \((x, y, t) \in \mathbb{R}^2 \times [0, \infty)\), and \( A, B, \lambda, \eta > 0 \)
   (c) \( u_t - u_x + 2u_y = e^{-x^2/2b^2} \cos(ay) \)
      with \( u = u(x, y, t) \), \((x, y, t) \in \mathbb{R}^2 \times [0, \infty)\), and \( a, b > 0 \)

Problems. Please write these up and turn them in.

1. Consider the PDE \( u(x) + \Delta^2 u(x) = f(x) \),
   where \( \Delta \) denotes the Laplacian operator (and \( \Delta^2 \) means applying the Laplacian twice), \( f : \mathbb{R}^d \to \mathbb{R} \) is given, and \( u : \mathbb{R}^d \to \mathbb{R} \) satisfies the boundary conditions \( u(x) \to 0 \) as \( |x| \to \infty \).
   You can assume that \( f \) is sufficiently nice that its Fourier transform is well-defined.
   (a) Find \( \hat{u}(\omega) \) in terms of \( \hat{f}(\omega) \).
   (b) Suppose \( f(x) = \delta(x) \) and \( d = 1 \). Is \( u \) continuously-differentiable as a function of \( x \)?
(c) Suppose \( f(x) = \delta(x) \). For what values of \( d \) do the functions \( u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_d} \) simultaneously belong to \( L^2(\mathbb{R}^d) \)?

2. Consider the wave equation in 3-D, i.e.

\[
 u_{tt} = \Delta u, \quad u : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R},
\]

with initial conditions

\[
 u(x, 0) = 0 \\
 u_t(x, 0) = \delta(x).
\]

Assume that \( u(x, t) \) decays sufficiently rapidly as \( |x| \to \infty \) so that its Fourier transform is well-defined.

(a) Find the Fourier representation of the solution, i.e., \( \hat{u}(\omega, t) \), \( \omega \in \mathbb{R}^3 \).

(b) Transform to spherical polar coordinates and show that

\[
 u(r, t) = \frac{1}{4\pi^2 r} \int_0^\infty \left[ \cos(\rho (r - t)) - \cos(\rho (r + t)) \right] d\rho,
\]

where \( r = |x| \) and \( \rho = |\omega| \).