Exercises. These do not need to be turned in.

1. Show that the Sturm-Liouville operator defined in class is self-adjoint.

2. Consider this example from class:

   \[ x^2 y''(x) + xy'(x) + y(x) = \mu y(x). \]  

   (a) By considering solutions of the form \( y(x) = c x^\alpha \), find the general solution of this equation. \textit{Hint: The general solution has the form} \( c_1 f_1(x) + c_2 f_2(x) \), \textit{where each of the} \( f_i \) \textit{is has the form above. You will find that the solutions are complex, however. By taking linear combinations, you can express every solution in the form} \( c_1' g_1(x) + c_2' g_2(x) \) \textit{where the} \( g_i(x) \) \textit{are real.}

   (b) Now suppose we want to solve the eigenvalue problem on the interval \( 1 \leq x \leq 2 \) with boundary conditions \( y(1) = y(2) = 0 \). Find the eigenvalues using the general solution above. (Remember we showed in class that \( \mu \leq 1 \).

3. Let \( H \) be the space of square-integrable functions on \([0, 1]\) such that \( u(0) = 0 \) and \( u'(0) = u(1) \), and let \( L \) be the operator \( \frac{d^2}{dx^2} \) acting on (a dense subset of) \( H \). Recall that the adjoint \( L^* \) is characterized by

   \[ \langle Lu, v \rangle = \langle u, L^* v \rangle \]  

   for all \( u \in D(L) \) and all \( v \in D(L^*) \), where \( D(L) \) is the domain of \( L \) (in this case, we can take it to be the space of twice-differentiable functions in \( H \)), and the domain \( D(L^*) \) of \( L^* \) is such that Eq. \( \text{(2)} \) holds. \( L \) is self-adjoint if \( L = L^* \) (and \( D(L) = D(L^*) \)).

   (a) Is \( L \) self-adjoint?

   (b) Consider the boundary value problem \( Lu = \lambda u \) for \( u \in H \). Show that the eigenvalue \( \lambda \) must satisfy \( \sqrt{\lambda} = \sin(\sqrt{\lambda}) \). Are there any real eigenvalues?

   (This exercise illustrates a subtlety in the notion of self-adjoint operators, namely that when we say \( "L = L^* \)," we assume implicitly that the two operators are defined on the same domain. There is a bit more discussion on this example in Ch. 6 of the notes.)
Problems. Please write these up and turn them in.

1. This problem explores a phase-plane view of the Sturm-Liouville problem (see Sect. 6.3.2 of the course notes for a more general formulation, but there is an error in Fig. 6.1).

Suppose \( u : [0, 1] \rightarrow \mathbb{R} \) is an eigenfunction for a regular Sturm-Liouville problem with Dirichlet boundary conditions \( u(0) = u(1) = 0 \). Let \( \lambda \) denote the corresponding eigenvalue, and let \( v(x) = p(x)u'(x) \).

(a) Using the function \( v \) above, we can rewrite the Sturm-Liouville equation as a pair of ODEs. Rewrite these ODEs in polar coordinates by letting \( u(x) = r(x) \sin(\theta(x)) \) and \( v(x) = r(x) \cos(\theta(x)) \) and deriving equations for \( \theta \) and \( r \).

(b) What conditions on \((r, \theta)\) correspond to the boundary conditions?

(c) Assuming \( q \) and \( \sigma > 0 \) are constants, show that \( \theta'(x) > 0 \) for all \( x \).

(d) Let’s order the eigenvalues so that \( \lambda_1 < \lambda_2 < \cdots \). Assuming \( \theta' \) is uniformly positive, sketch what phase plane curves corresponding to \( \phi_1, \phi_2, \) and \( \phi_3 \) should look like. Also sketch the corresponding graphs as functions of \( x \).

(e) How many zeros does \( \phi_n \) have inside \((0, 1)\)? How are the zeros of \( \phi_n \) related to those of \( \phi_{n+1} \)?

Note: This should give you a hint as to why the eigenvalues do not accumulate except at \( \infty \).

2. For the inhomogeneous equation

\[
x^2y''(x) + xy'(x) + y(x) = f(x), \quad 1 \leq x \leq 2, \quad y(1) = y(2) = 0,
\]

write down a general expression for the solution as an orthogonal expansion using the eigenfunctions of Eq. (I).