1. Let $f$ be the function on $\mathbb{R}$ defined by $f(x) = x^2 + x$ for all real numbers $x$.

Your proofs should use only simple algebra and definitions; do not use limits or derivatives, or properties of continuous functions or polynomials or quadratic functions or quadratic equations or parabolas.

a) Find the image of the interval $[-1, 0]$ under this function, and prove your answer clearly, carefully, and completely.

**SOLUTION.** Note that, for each $x$, $f(x) = x^2 + x = (x + 1/2)^2 - 1/4$. From this, we can see, perhaps with the help of a graph, that the image $f([-1, 0])$ is the interval $[-1/4, 0]$. Here is the proof:

**First, consider an element $y$ of $f([-1, 0])$.** Then, by definition of image, we can find $x$ in $[-1, 0]$ such that $y = f(x) = x^2 + x = (x + 1/2)^2 - 1/4$. We use this formula to prove, algebraically, that $y$ is in $[-1/4, 0]$.

From the fact that $y = (x + 1/2)^2 - 1/4$, we see that $y \geq -1/4$, because $(x + 1/2)^2 \geq 0$.

Since $-1 \leq x \leq 0$, we have $-x \geq x^2$, since multiplying by the negative number $x$ reverses the inequality. Thus, $0 \geq x^2 + x$. So $y = x^2 + x \leq 0$. **This shows that $y$ is in $[-1/4, 0]$**.

Conversely, consider $y$ in $[-1/4, 0]$. We want to show that there exists $x$ in $[-1, 0]$ such that $y = f(x)$. Then $y \geq -1/4$, so $(y + 1/4)^{1/2}$ exists. We choose $x = (y + 1/4)^{1/2} - 1/2$. **We need to show that $x$ is in the interval $[-1, 0]$ and that $y = f(x)$**.

First, since $-1/4 \leq y \leq 0$, we have

$$(-1/4 + 1/4)^{1/2} - 1/2 \leq (y + 1/4)^{1/2} - 1/2 \leq (0 + 1/4)^{1/2} - 1/2 = 0,$$

so $-1/2 \leq x \leq 0$. Thus, $x$ is in $[-1, 0]$.

Next, $f(x) = x^2 + x = [(y + 1/4)^{1/2} - 1/2]^2 + [(y + 1/4)^{1/2} - 1/2]$

$$= (y + 1/4) - (y + 1/4)^{1/2} - 1/4 + (y + 1/4)^{1/2} - 1/2 = y,$$

as desired. (This calculation would be much simpler, and equally correct, if we used the alternative form $f(x) = (x + 1/2)^2 - 1/4$, but the approach above is “safer” since it uses directly the original form of the function to show that $f(x) = y$.)

**This proves that $y$ is in the image $f([-1, 0])$**.

Since every element of $f([-1, 0])$ is in $[-1/4, 0]$, and every element of $[-1/4, 0]$ is in $f([0, 1])$, we conclude that $f([-1, 0]) = [-1/4, 0]$.
b) Let $g$ be the function defined by the same formula, with domain $[0, \infty)$.

Prove or disprove: $g$ is injective. (As usual, use the standard approach discussed in class.)

**SOLUTION.** We will prove that $g$ is injective. Consider nonnegative numbers $a$ and $b$ such that $g(a) = g(b)$. Then, since $g(x) = (x + \frac{1}{2})^2 - \frac{1}{4}$, we have

$$(a + \frac{1}{2})^2 - \frac{1}{4} = (b + \frac{1}{2})^2 - \frac{1}{4}.$$  

From here, we see that $(a + \frac{1}{2})^2 = (b + \frac{1}{2})^2$, so $|a + \frac{1}{2}| = |b + \frac{1}{2}|$.

**Since $a$ and $b$ are nonnegative,** $a + \frac{1}{2} = b + \frac{1}{2}$, so $a = b$.

We have shown that if $g(a) = g(b)$, then $a = b$. This proves that $g$ is injective.