1. Let f be the function on $\mathbb{R}$ defined by $f(x) = x^2 + x$ for all real numbers x.

Your proofs should use only **simple algebra** and definitions; do not use limits or derivatives, or properties of continuous functions or polynomials or quadratic functions or quadratic equations or parabolas.

a) Find the image of the interval $[-1, 0]$ under this function, and prove your answer clearly, carefully, and completely.

SOLUTION. Note that, for each x, $f(x) = x^2 + x = (x + {}^1/_2)^2 - {}^1/_4$. From this, we can see, perhaps with the help of a graph, that the image $f([-1, 0])$ is the interval $\mathbf{[-{}^1/_4\,,\, 0]}$. Here is the proof:

First, consider an element y of $f([-1, 0])$. Then, by definition of image, we can find x in $[-1, 0]$ such that $y = f(x) = x^2 + x = (x + {}^1/_2)^2 - {}^1/_4$. We use this formula to prove, algebraically, that y is in $[-{}^1/_4, 0]$.

From the fact that $y = (x + {}^1/_2)^2 - {}^1/_4$, we see that $y \geq -{}^1/_4$, because $(x + {}^1/_2)^2 \geq 0$.

Since $-1 \leq x \leq 0$, we have $-x \geq x^2$, since multiplying by the negative number x reverses the inequality. Thus, $0 \geq x^2 + x$. So $y = x^2 + x \leq 0$. **This shows that y is in $[-{}^1/_4, 0]$.**

Conversely, consider y in $[-{}^1/_4, 0]$. We want to show that there exists x in $[-1, 0]$ such that $y = f(x)$. Then $y \geq -{}^1/_4$, so $(y + {}^1/_4)^{1/2}$ exists. We choose $x = (y + {}^1/_4)^{1/2} - {}^1/_2$. **We need to show that x is in the interval $[-1, 0]$ and that $y = f(x)$.**

First, since $-{}^1/_4 \leq y \leq 0$, we have

$$(-{}^1/_4 + {}^1/_4)^{1/2} - {}^1/_2 \;\leq\; (y + {}^1/_4)^{1/2} - {}^1/_2 \;\leq\; (0 + {}^1/_4)^{1/2} - {}^1/_2 \;=\; 0,$$

so $-{}^1/_2 \leq x \leq 0$. Thus, x is in $[-1, 0]$.

Next, $f(x) \;=\; x^2 + x \;=\; [(y + {}^1/_4)^{1/2} - {}^1/_2\,]^2 + [(y + {}^1/_4)^{1/2} - {}^1/_2\,]$

$$= (y + {}^1/_4) - (y + {}^1/_4)^{1/2} + {}^1/_4 + (y + {}^1/_4)^{1/2} - {}^1/_2 \;=\; y, \text{ as desired.}$$

(This calculation would be much simpler, and equally correct, if we used the alternative form $f(x) = (x + {}^1/_2)^2 - {}^1/_4$, but the approach above is "safer" since it uses directly the original form of the function to show that $f(x) = y$.)

This proves that y is in the image $f([-1, 0])$.

Since every element of $f([-1, 0])$ is in $[-{}^1/_4, 0]$, and every element of $[-{}^1/_4, 0]$ is in $f([0, 1])$, we conclude that $f([-1, 0]) = [-{}^1/_4, 0]$.

b) Let g be the function defined by the same formula, with domain $[0, \infty)$.
Prove or disprove: g is injective. (As usual, use the standard approach discussed in class.)

SOLUTION. We will prove that ***g* is injective**. Consider nonnegative numbers a and b such that $g(a) = g(b)$. Then, since $g(x) = (x + 1/2)^2 - 1/4$, we have

$$(a + 1/2)^2 - 1/4 = (b + 1/2)^2 - 1/4.$$

From here, we see that $(a + 1/2)^2 = (b + 1/2)^2$, so $|a + 1/2| = |b + 1/2|$.
Since *a* and *b* are nonnegative, $a + 1/2 = b + 1/2$, so $a = b$.

We have shown that if , then $a = b$. This proves that g is injective.