2. Suppose f is an injective function. Suppose $x \in \text{dom}(f)$ and $C \subseteq \text{dom}(f)$.
Prove or disprove:

$$\text{If } f(x) \in f(C), \text{ then } x \in C.$$

In either case, give a clear, careful, correct **symbolic** proof, which explicitly uses the **definitions** of image and injective, not just a verbal "proof" which explains why the result is true.

SOLUTION. Suppose f is injective and $f(x) \in f(C)$. Then, by definition of image, we can find c in C such that $f(x) = f(c)$. Since f is injective, $x = c$. Since $c \in C$, $x \in C$.

3. Prove or disprove the following statement:

$$\text{If } f \text{ and } C \text{ are as in the preceding problem, then } f^{\leftarrow}(f(C)) = C.$$

In either case, give a clear, careful, correct proof, using the standard approach discussed in this class for proving or disproving such statements.

SOLUTION. Suppose f is injective. First, consider x in C. Then, by definition of image, $f(x) \in f(C)$, so by definition of pre-image, $x \in f^{\leftarrow}(f(C))$.
Conversely, consider x in $f^{\leftarrow}(f(C))$. Then, by definition of pre-image, $f(x) \in f(C)$. Since f is injective, we conclude from Problem 2 that $x \in C$.
We have shown that every element of C is element of $f^{\leftarrow}(f(C))$, and every element of $f^{\leftarrow}(f(C))$ is an element of C. This prove that $f^{\leftarrow}(f(C)) = C$, as desired.

4. a) Give the definitions of "finite set" and "denumerable set" in a way which could be understood by someone who does not know what "equinumerous" means. (Use functions, not the word "equinumerous".) E.g., a set S is finite iff ... .

SOLUTION. A set S is finite iff either $S = \varnothing$ or there exists a natural number n and a bijection from S onto $[0, n] \cap \mathbb{N}$.
A set S is denumerable iff there exists a bijection from S onto $\mathbb{N}$,
or, equivalently, a bijection from $\mathbb{N}$ onto S.

b) Prove that the set of even natural numbers is denumerable.

SOLUTION. Let $\mathbb{E}$ be the set of even natural numbers. Define a function f by

$$\text{for each } n \text{ in } \mathbb{N}, \ f(n) = 2n.$$

We will provae that f is a bijection from $\mathbb{N}$ onto $\mathbb{E}$.
Clearly, $f(n)$ is defined for each n in $\mathbb{N}$, and $f(n)$ is an even natural number, so $f : \mathbb{N} \to \mathbb{E}$.
f is injective, because if $f(n) = f(m)$ for some n and m in $\mathbb{N}$, then $2n = 2m$, so $n = m$.
To prove that $f : \mathbb{N} \to \mathbb{E}$ is surjective, consider n in $\mathbb{E}$. Then, by definition of even number, we can find k in $\mathbb{N}$ such that $n = 2k$. Clearly, $f(k) = 2k = n$.
So for every n in $\mathbb{E}$ there eixsts k in $\mathbb{N}$ such that $f(k) = n$. Thus, $f : \mathbb{N} \to \mathbb{E}$ is surjective.
This proves that f is a bijection from $\mathbb{N}$ onto $\mathbb{E}$, so $\mathbb{E}$ is denumerable.

5. Classify the following sets as finite (F), denumerable (D), countable (C), and/or uncountable(U).

SOLUTION.

a. Ø **F, C**

b. $\mathbb{Z}$ (the integers) **D, C**

c. $\mathbb{Q}$ (the rationals) **D, C**

d. $\mathbb{R}$ (the reals) **U**

e. $\mathbb{R} \setminus \mathbb{Q}$ (the irrationals) **U**

f. F = the set of all students who have failed Math 323 at UA. **F, C**.

6. We say that a subset S of $\mathbb{R}$ is a **simpleton** iff

for all x in S, there exists $m > 0$ such that $x > m$.

a) Consider the following assertion:

Assertion(a). The following is a general description of a simpleton:

A subset S of $\mathbb{R}$ is a simpleton iff $S = (0, \infty)$.

Comment on the meaningfulness and validity the following sentences as a way of starting the proof of this assertion:

Suppose x is in S. Then $x > m$.

SOLUTION. Who is S? Who is m? These variables are not defined or introduced.

b) Prove or disprove the assertion in Part (a).

SOLUTION. The assertion is not true because, e.g., the set $\{1\}$ is easily shown to be a simpleton but it is not $(0, \infty)$.

(To prove that $\{1\}$ is a simpleton, consider x in $\{1\}$. Then $x = 1$. Let $m = 1/2$. Then $m > 0$ and $x > m$. Thus, for all x in $\{1\}$, there exists $m > 0$ such that $x > m$, so $\{1\}$ is a simpleton.)

c) Consider the following assertion:

Assertion(c). The following is a general description of a simpleton:

A subset S of $\mathbb{R}$ is a simpleton iff S is the interval (m, ∞).

Comment on the meaningfulness and validity of this assertion.

SOLUTION. The assertion is not meaningful because m has not been introduced.

If m were introduced, **the assertion is not valid** because, as we saw in part (b), there are simpletons which do not look like intervals of the from $[m, \infty)$.