

1. Example.

Throughout the following, be sure to use function notation correctly in your solutions! In particular, distinguish between the name of a function (such as f) and the output of such a function (such as $f(x)$).

Let S be the set of all real-valued functions on the set of real numbers which have derivatives of all orders: I.e., the first derivative exists, the second derivative exists, etc. Define the function F as follows: For each g in S ,

$$F(g) = \{f \text{ in } S : f' = g\}.$$

COMMENT. Note that the set $F(g)$ is simply the set of all antiderivatives of the function g .

(a) Explain why F is a function from S to the power set $\mathcal{P}(S)$.

Note. This is “obvious”, but give an answer with enough explanation so that it is clear to your reader that you know enough about sets and functions to know why it is obvious.

SOLUTION. $F(g)$ is defined for every element g of S , and $F(g)$ is defined to be a set which consists of elements of S , so $F(g)$ is an element of the power set of S . Thus, $F : S \rightarrow \mathcal{P}(S)$.

(b) Let id be the identity function on the set of real numbers. Using your knowledge of calculus, determine if id is in S . (Formal proof not needed; just explain.) If it is, what is $F(id)$ in this case? (Don't just give the definition of $F(id)$! As usual when asked such a question, tell what $F(id)$ is for this particular case.)

SOLUTION. The function id is defined by $id(x) = x$ for every real number x . From calculus, we know that the first derivative is the constant function 1 and the second derivative and all higher derivatives are the constant function 0. Thus, id has derivatives of all orders.

COMMENT: It is a very serious mistake to think that because a derivative is 0, then the derivative does not exist! (Similarly, the fact that a set is empty does not mean that the set does not exist!)

Also from calculus we know that the set of all antiderivatives of the identity function is the set of all functions of the form $(1/2)x^2 + C$, where C is an arbitrary constant. More precisely,

$$F(id) = \{f \text{ in } S : \exists \text{ real } C \ni \forall \text{ real } x, f(x) = (1/2)x^2 + C\}$$

(c) Using your knowledge of calculus (and linear algebra and differential equations), describe all functions g which have the property that g is an element of $F(g)$.

Note. Do this clearly and precisely, either by using set notation:

$$\{g : g \text{ is an element of } F(g)\} = \{g : g \in F(g)\} = \{g : \dots\}$$

or (probably better) by using an “iff” statement such as

$$g \in F(g) \text{ iff } \dots$$

Describe this set of functions clearly and precisely, and explain. Again, the point is not to repeat the general definition of $F(g)$, but to describe specifically the functions which satisfy the stated condition.

SOLUTION. (BRIEF) From calculus, $\{g : g \in F(g)\} = \{g : \exists \text{ real } C \ni \forall \text{ real } x, g(x) = Ce^x\}$,
or $g \in F(g) \text{ iff } \exists \text{ real } C \ni \forall \text{ real } x, g(x) = Ce^x$.

(d) What is the set N in this case? (This refers to the set N in Problem 2 on the “main page” of the Project.) Describe clearly and precisely, so that someone looking at your description of the set could tell what is in the set (and what is not in the set) without knowing anything about the other problems in this project.

SOLUTION. Roughly speaking, the set N (i.e., the set of all g such that $g \notin F(g)$) is the set of all “non-base- e exponential functions”, i.e.,

$$N = \{g : \text{there does not exist real } C \ni \forall \text{ real } x, g(x) = Ce^x\}$$

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2. Example.

Let R be an equivalence relation on a nonempty set S . For each a in S , let $F(a)$ denote the equivalence class of a determined by the equivalence relation R .

(a) Explain why F is a function from S to $\mathcal{P}(S)$.

SOLUTION. Every element of S has an equivalence class which is a subset of S . So, $F(a)$ is defined for all a in S and is an element of the power set of S . Thus, $F : S \rightarrow \mathcal{P}(S)$.

(b) What elements of S have the property that a is an element of $F(a)$?

SOLUTION. Every element of S is in the equivalence class of itself.

(c) What is the set N in this case?

SOLUTION. Since every element a of S satisfies $a \in F(a)$, the set N is empty.

3. Example.

Let S be a nonempty set. For each a in S , let $F(a) = S \setminus \{a\}$.

(a) Explain why F is a function from S to $\mathcal{P}(S)$.

SOLUTION. $F(a)$ is defined for every element of S and is, by definition, a subset of S .

(b) What elements of S have the property that a is an element of $F(a)$?

SOLUTION. From the way that $F(a)$ is defined, no element a is an element of $F(a)$.

(c) What is the set N in this case?

SOLUTION. Since the statement $a \notin F(a)$ is true for every element a of S , $N = S$.

NOTE THAT IT FOLLOWS FROM EXAMPLES 1 AND 3 THAT, IN GENERAL, THE SET N IS NOT EMPTY.

Hence, it does not make sense in Problem 4 in the main part of the project to say that N is empty. Obviously, from these examples, this is not generally true.