

Suppose f is a real-valued continuous function defined on a closed interval $[a, b] \subseteq \mathbb{R}$. Suppose $f(a) \neq f(b)$, and let y be a number between $f(a)$ and $f(b)$. Then there exists a number c in (a, b) such that $f(c) = y$.

Proof (using the “bisection method” and the Nested Intervals Theorem).

For definiteness, we assume that $f(a) < y < f(b)$. The proof in the other case is similar.

To use the Nested Intervals Theorem, we need a sequence of closed, nested intervals, and as a start in this direction we define $a_1 = a$ and $b_1 = b$. So

$$f(a_1) < y < f(b_1).$$

Let c_1 be the midpoint of the interval $[a_1, b_1]$. There are three possibilities:

If $f(c_1) = y$, we are done. Otherwise, we continue ...

If $f(c_1) > y$, choose $b_2 = c_1$, so $f(b_2) > y$, and choose $a_2 = a_1$.

[DRAW A PICTURE!]

If $f(c_1) < y$, choose $a_2 = c_1$, so $f(a_2) < y$, and choose $b_2 = b_1$.

[DRAW A PICTURE!]

In either case,

$$f(a_2) < y < f(b_2).$$

Now we repeat the procedure for the interval $[a_2, b_2]$.

Let c_2 be the midpoint of the interval $[a_2, b_2]$. There are three possibilities:

If $f(c_2) = y$, we are done. Otherwise, we continue ...

If $f(c_2) > y$, choose $b_3 = c_2$, so $f(b_3) > y$, and choose $a_3 = a_2$.

If $f(c_2) < y$, choose $a_3 = c_2$, so $f(a_3) < y$, and choose $b_3 = b_2$.

In either case,

$$f(a_3) < y < f(b_3).$$

Continue in this way, if necessary. If we get to the n -th stage, we have an interval $[a_n, b_n]$ such that

$$f(a_n) < y < f(b_n).$$

Let c_n be the midpoint of the interval $[a_n, b_n]$. There are three possibilities:

If $f(c_n) = y$, we are done. Otherwise, we continue ...

If $f(c_n) > y$, choose $b_{n+1} = c_n$, so $f(b_{n+1}) > y$, and choose $a_{n+1} = a_n$.

If $f(c_n) < y$, choose $a_{n+1} = c_n$, so $f(a_{n+1}) < y$, and choose $b_{n+1} = b_n$.

In either case,

$$f(a_{n+1}) < y < f(b_{n+1}).$$

Thus, either the process terminates when f has the desired value y at some midpoint of one of our intervals, or we get a nested sequence of closed intervals $\{[a_n, b_n]\}$ with the property that for every n , $f(a_n) < y < f(b_n)$. Furthermore, since each term in the sequence is an interval of half the length of the preceding term in the sequence, $\{b_n - a_n\}$ converges to 0. Thus, by the Nested Interval Theorem, these sequences have an intersection which consists of one point, say c , with the property that $\{a_n\}$ converges to c and $\{b_n\}$ converges to c . By the continuity of f , $\{f(a_n)\}$ converges to $f(c)$ and $\{f(b_n)\}$ converges to $f(c)$. From the inequalities above,

$$f(c) \leq y \leq f(c),$$

and so $f(c) = y$, as desired.