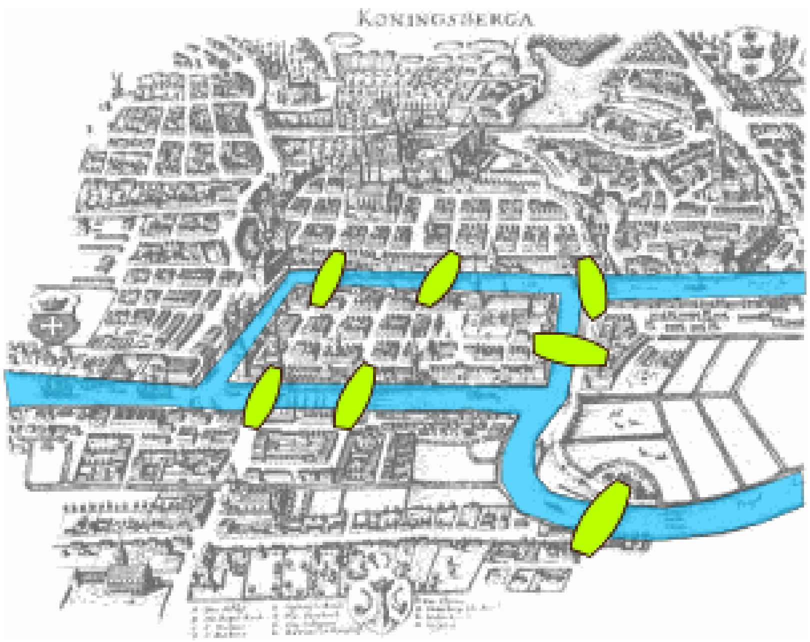
THE KÖNIGSBERG BRIDGE PROBLEM

MATH IN MODERN SOCIETY - MATH 105 - FALL 05 - INSTRUCTOR TOM LAGATTA

1. Introduction to the Problem

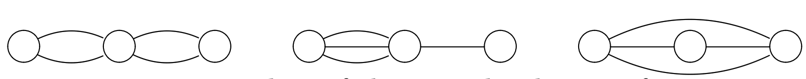
The Königsberg Bridge Problem is a classic problem, based on the topography of the city of Königsberg, Prussia. The city sits along the river Pregel, in which there are two large islands. Historically, seven bridges connected the islands to the mainland, as depected in the following image.



The problem is thus: Starting at either the mainland or one of the islands, can one follow a route which crosses every bridge *exactly* once? Moreover, if such routes exists, is there one which begins and ends at the same place? A solution to this problem must take one of two forms: either an *example* of such a route, or a *proof* of why none exists. The famous Swiss mathematician Leonhard Euler (pronouned "Oiler") considered and solved this problem in 1736.

2. Basic Definitions of Graph Theory

Before we discuss Euler's solution to the problem, let us discuss the language of *graph theory*. A *graph* is a collection of *vertices* (also called nodes or points), and *edges* (also called lines). The following three examples are all graphs with 3 vertices and 4 edges:



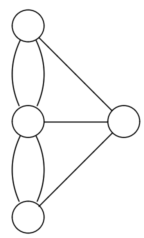
Another important concept is that of *degree*. The degree of a vertex is the number of edges that meet the vertex (counting loops twice). In the example above,

- The first graph has 2 vertices of degree 2, and 1 vertex of degree 4;
- The second graph has 1 vertex of degree 3, 1 vertex of degree 4, and 1 vertex of degree 1; and
- The third graph has 2 vertices of degree 3, and 1 vertex of degree 2.

A few more definitions: A *path* on a graph is exactly what we expect it to be: a way to get from one vertex to another., and a *circuit* is a path which ends at the vertex where it began. An *Eulerian path* or *Eulerian circuit* is a path or circuit which traverses every edge of the graph exactly once. Notice that every Eulerian circuit is a Eulerian path, though not vice versa. This name of course is an homage to Euler, who is considered the father of graph theory.

3. Euler's Solution

Now that we have some definitions under our belt, let us reformulate the Königsberg problem in terms of graph theory:



Here, the top and bottom vertices represent the mainland, and the middle two represent the islands. Our problem asks, "Does the Königsberg graph have an Eulerian path or circuit?"

Now we state Euler's magnificent result, the theorem that put graph theory in the mathematical spotlight:

Theorem 1.

(a) If a graph has 0 odd-degree vertices, then it has at least one Eulerian circuit.

(b) If a graph has 2 odd-degree vertices, then it has at least one Eulerian path, but no circuit.

(c) If a graph has more than 2 odd-degree vertices, it has neither.

Proof. It is not possible to have a graph with 1 odd-degree vertex (this will be a homework problem I assign).

Suppose we walk along an Eulerian path, and encounter an odd-degree vertex (not the start). If it has just degree 1, then that must be the edge we arrive from. We have no place to go, so it must be the endpoint. If it has degree 3, then we arrive, leave, and arrive again. Again, we are stuck, and it must be the endpoint. If if has degree 5, then we arrive, leave, arrive, leave, and arrive again. We can make the same argument for any odd-degree vertex. This shows that any graph with more than 2 odd-degree vertices cannot have an Eulerian path (nor circuit).

Now, to construct an Eulerian path or circuit, we use *Fleury's algorithm*:

1. Pick a vertex as the starting point. (If there are odd-degree vertices, choose one of these. Otherwise, pick any vertex.)
2. Whenever you have a choice, always choose to travel along an edge that does not cut off part of the graph.

3. Label the edges in the order in which you travel them.
4. When you can't travel any more, stop. You are done!

□

Now, let us apply this theorem to the Königsberg problem. Counting degrees, we see that the Königsberg graph has 3 vertices of degree 3, and 1 vertex of degree 5. Since there are more than two vertices of odd degree, we know that the graph has no Eulerian path! Violà, this

Graphs are quite useful in many situations, from real world applications to pure mathematics, though we will only scratch the surface of their potential. I hope this short introduction on graph theory was enough to whet your appetite. For more information, I recommend reading chapters 5-7 of the textbook, or reading

`http://en.wikipedia.org/wiki/Graph_theory`.