HOMEWORK 8
DUE WEDNESDAY, 12 MARCH 2008

MATH 215 - LINEAR ALGEBRA - TOM LAGATTA

I want to add something to today's lecture, connecting matrix algebra back to real algebra. Solving the real equation $ax = b$ is easy for any real number, as long as $a \neq 0$. Solving the matrix equation $A\vec{x} = \vec{b}$ is exactly the same procedure, only here the condition $A \neq O$ must be relaxed to the condition that A is invertible:

$$\begin{aligned} ax &= b \\ a^{-1}(ax) &= a^{-1}b \\ x &= a^{-1}b \end{aligned} \qquad \begin{aligned} A\vec{x} &= \vec{b} \\ A^{-1}(A\vec{x}) &= A^{-1}\vec{b} \\ \vec{x} &= A^{-1}\vec{b} \end{aligned}$$

On the past two homeworks, I've mentioned the non-invertible matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

As you saw on the last homework, MATLAB lets you "invert" this matrix even though it's not invertible. Let me explain why by another analogy with real numbers, where the only non-invertible real number is $a = 0$. We can't solve the equation $0x = 3$, since there is division by zero is forbidden. But computers doesn't see much difference in the equations

$$0x = 3 \qquad \text{or} \qquad 10^{-13}x = 3,$$

since there is some rounding error, and there *is* a solution to the second equation. While MATLAB is smart enough to know the difference between 0 and 10^{-13}, it doesn't know the difference between the matrix A above and a slightly modified one.[1]

Recall that I asked you in Section 2.3, problem 8, to determine if $\vec{b} = [10, 11, 12]$ was in the span of the columns of A. This is equivalent to solving the matrix equation

$$A\vec{x} = \vec{b}.$$

You found using row reduction that the solution set was

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t - 9.3333 \\ -2t + 9.6667 \\ t \end{bmatrix},$$

where $z = t$ was a free variable, so there were infinitely many solutions. If A had been invertible in the first place, then $\vec{x} = A^{-1}\vec{b}$ would have been a *unique* solution.

Date: Spring 2008.

[1]Or maybe it does know the difference, and just assumes that no self-respecting linear algebra instructor would ask his students to invert a non-invertible matrix.

- Read pages 152-176 in the textbook (Sections 3.2 and 3.3). The exam will cover up through page 162. This includes Example 3.23 and Theorem 3.6, but *not* Theorem 3.7.

- **Section 3.2:** 1, 2, 3, 4, 19, 20, 21, 23, 24, 25, 26, 28, 34, 35, 36

- **Sections 3.2 and 3.3:**
  (1) Prove that
  $$A\left[\vec{b}_1|\cdots|\vec{b}_n\right] = \left[A\vec{b}_1|\cdots|A\vec{b}_n\right]$$
  by looking at the ij^{th} entry on both sides.
  (2) Show by direct, hand-calculation that the inverse of
  $$A = \begin{bmatrix} 10 & 4 \\ 4 & 2 \end{bmatrix} \qquad \text{is} \qquad A^{-1} = \begin{bmatrix} .5 & -1 \\ -1 & 2.5 \end{bmatrix}.$$
  (3) Mimicking Example 3.23, show that the matrix
  $$C = \begin{bmatrix} -3 & 1 \\ 6 & 2 \end{bmatrix}$$
  is not invertible.

- Notes:
  - Notice that I have not asked you to prove Theorem 3.3(a), that $A(BC) = (AB)C$ (associativity). This proof is worth solving through once in your life, because you will learn a lot about matrix manipulations, getting your hands dirty, and building character. Unfortunately, Amendment VIII of the United States Constitution legally prohibits me from assigning it to you.

- **Exam Review:**
  - You should be an expert on Gaussian elimination. As we've seen thus far, the technique for most problems in chapters 2 and 3 is:
    1. Interpret the problem as a system of linear equations,
    2. Row reduce with Gaussian elimination, and
    3. Conclude something about your problem.

    Most of you don't have much trouble with (1). For (2), it is important to go back and check your work. If you have arithmetic errors in your Gaussian elimination, you will lose lots of points. For (3), I *never* want an answer in the form of a row-reduced matrix. You should always interpret this in the context of the question given.

  - Reprove Theorems 2.5, 2.6, 2.7, 2.8, 3.1(b), 3.2, 3.3(not part a), 3.4, 3.5, 3.6, and 3.7. You should rework through most of these in full detail, without looking at the book. After you've written your proof up, ask a classmate (or me) to look it over. The only good way to learn proofs is to do proofs, and the benefit of reproving textbook theorems is that you have a finished version to compare after you're done.

  - Restate the rank theorem. Prove the following corollary:
    If A is an $m \times n$ matrix and $r = rank(A)$, then the columns of A span an r-dimensional flat space sitting inside $\mathbb{R}^m$.

    Hint: Consider the matrix system $A\vec{c} = \vec{b}$ for $\vec{c} \in \mathbb{R}^n$ and $\vec{b} \in \mathbb{R}^m$. What are the significances of $\vec{c}$ and $\vec{b}$? How are their sizes relevant? How is the "number of free variables" stated in the theorem related to the dimension of the span of the columns?