Continuity & Differentiability

(continued)

if \[ \lim_{\delta \to 0} |f(x) - f(x_0 + \delta)| = 0 \]
then \( f \) is continuous at \( x_0 \)

if \[ \lim_{\delta \to 0} \left[ \frac{f(x_0 + \delta) - f(x_0)}{\delta} \right] \text{ exists} \quad (= f'(x_0)) \]
then \( f \) is differentiable at \( x_0 \)

Look at \[ \frac{f(x)}{x} = \frac{x}{x} = 1 \]

Look at \[ \left| \frac{f(x)}{x^2} \right| \to \infty \quad \text{as} \quad x \to 0 \]

Ex. Take \( f(x) = x \)
As \( x \to 0 \), \( f(x) \to 0 \)

Take \( f(x) = x^2 \)
As \( x \to 0 \), \( f(x) \to 0 \)

\[ y \]

\[ f(x) \]

\[ x \]

\[ x_0 \]

\[ x_0 + \delta \]

\[ x \]
Mean-value theorem: If $f$ is (continuous and differentiable on $[a, b]$, then there exists a value $c$, $a \leq c \leq b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

Local & Global Extrema

Definition: A point $x_0$ is a global maximum of a function $f$ if $f(x_0) \geq f(x)$ for every $x$ in the domain of definition of $f$.

It is possible for a function to have a local minimum at $x=0$ but not have $f'(x) = 0$.

Ex: $|x|$, $f(x) = x$ on $[0, 1]$. 

It is possible to have \( f'(x_0) = 0 \) and \( x_0 \) not being a maximum or local minimum of \( f \).

\[
\text{Ex: } f(x) = x^3 \quad \text{at } x = 0
\]
\[
f'(x) = 3x^2 \quad f'(0) = 0
\]

\[
y = f(x) = x^3
\]

\[\begin{array}{c}
\text{It is possible for } f''(x_0) \text{ to be 0 and for } f \text{ not to have an inflection point at } x_0 \\
\text{Ex: } f(x) = x^4 \text{ at } x = 0
\end{array}\]
\[
f'(x) = 4x^3 \quad f''(x) = 12x^2 \quad f''(0) = 0\]