Introduction to differential equations

1. A simple differential equation

- $f(x) = e^x$ is such that $f'(x) = f(x)$
- Is it the only function that has this property?

To answer this, we want to solve

$$\frac{dy}{dx} = y$$ for an unknown function y.

If $y \neq 0$, one has

$$\frac{1}{y}\frac{dy}{dx} = 1$$

Integrate both sides with respect to x:

$$\int \frac{1}{y}\frac{dy}{dx}\,dx = \int dx = x + c$$

don't forget the constant!

$$\Rightarrow \ln|y| = x + c$$

take the exponential

$$\Rightarrow |y| = e^{x+c}$$

$$\Rightarrow y = \pm e^{x+c} = \pm e^x e^c = (\pm e^c) e^x = A e^x$$

Therefore

$$y(x) = K e^{x} \quad \text{is a } \underline{\text{family}}$$

of solutions to $y' = y$, Here $K \in \mathbb{R}$.

<u>Definitions</u>:

- An <u>ordinary differential equation</u> (ODE) is an equation that involves at least 1 derivative of an unknown function y, and may relate y and its derivatives to the independent variable x.

<u>Ex</u>: $\dfrac{dy}{dx} = y$

dependent variable

x: independent variable

- The <u>order</u> of a differential equation is the order of the highest derivative in the equation.

  <u>Ex</u>: $y'' + 2y' + 3y = x$ is 2^{nd} order

- If the <u>independent variable</u> does not appear explicitly in the ode, one says that the differential equation is <u>autonomous</u>.

- An <u>ode</u> is linear if the terms involving the <u>dependent variable</u> are all <u>linear</u>.

  <u>Ex</u>: $y'' + 2y' + 3y^2 = x$ non-linear

  $y'' + 2y' + 3y = \cos(x)$ linear

- An <u>explicit solution</u> to an ode is a function (e.g. $y(x)$) which when substituted into the ode is such that both sides of the equal sign are the same.

<u>Ex.</u> $y' = y$ $\quad$ $y(x) = e^x$ is an explicit solution of this ode.

<u>To check:</u> $y(x) = e^x$ $\quad$ $y'(x) = e^x = y$ ✓

- Typically, you have a family of solutions, parametrized by a number of arbitrary constants.

<u>Ex.</u> $y(x) = K e^x$ solves $y' = y$.

- An <u>initial condition</u> specifies the value(s) of $y, y', \dots,$ at $x = 0$.

Note that for a 1st order equation, it is enough to specify $y(0)$ in order to pick the arbitrary constant.

- A <u>boundary condition</u> defines the value of y at some point which is not the initial point.

<u>Ex.</u> $y(x) = K e^x$

<u>Initial condition</u> : $y(0) = 1$

$$1 = y(0) = K e^0 = K \quad \text{so} \quad y(x) = K e^x = e^x$$

<u>Boundary condition</u> : $y(1) = e$

$$e = y(1) = K e^1 = K e \Rightarrow K = 1$$

So $y(x) = e^x$.