Equations of the form $\frac{dy}{dx} = g(x)$

$$\frac{dy}{dx} = g(x)$$

Integrate both sides

$$y(x) = \int g(x)\,dx + C$$

Fundamental theorem of Calculus:

if f is continuous on the $[a,b]$ and if $F'(x) = f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) = \left[F(x)\right]_a^b$$

Second fundamental theorem of Calculus

If f is continuous on $[a,b]$ and if $x \in [a,b]$, then $F(x) = \int_a^x f(t)\,dt$ is an antiderivative of f, i.e.

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Going back to the differential equation

$$y' = g(x)$$

we have a solution in the form

$$y(x) = \int_a^x g(t)\,dt + c$$

This general form is called the <u>general solution</u> of the ode.

If we have an initial or boundary condition of the form

$y(x_0) = y_0$, then we pick 1 solution from the whole family:

$$y(x) = \int_{x_0}^x g(t)\,dt + y_0$$

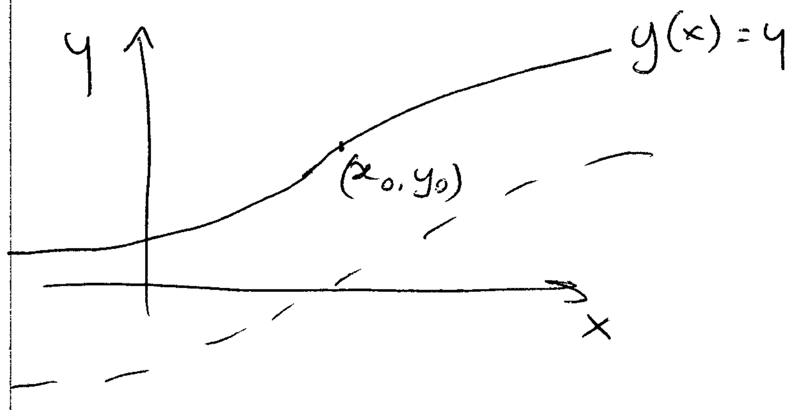
This is a <u>particular solution</u>, which satisfies $y(x_0) = y_0$.

What does uniqueness have to do with the crossing of solution curves?

We start with $y' = g(x)$ and a point (x_0, y_0).

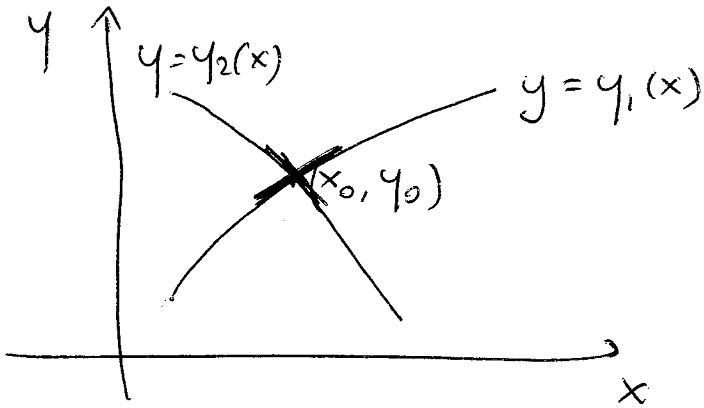
1. Existence: we ask: can we find a solution curve that goes through (x_0, y_0)?

Graphically:



Is there a $y(x)$ such that $y(x_0) = y_0$ and $y'(x) = g(x)$?

2. Uniqueness: now that we know we can find one solution, we ask: is it possible to have more than 1 solution going through a given point?

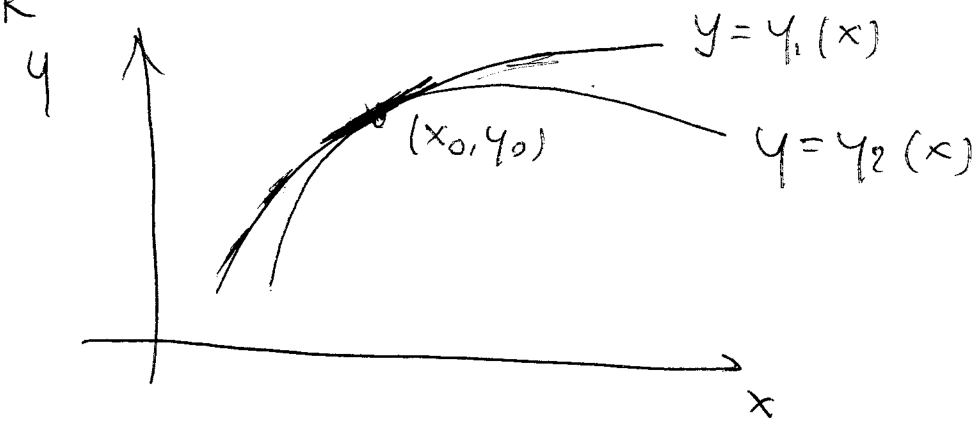


$$y_1'(x) = g(x)$$

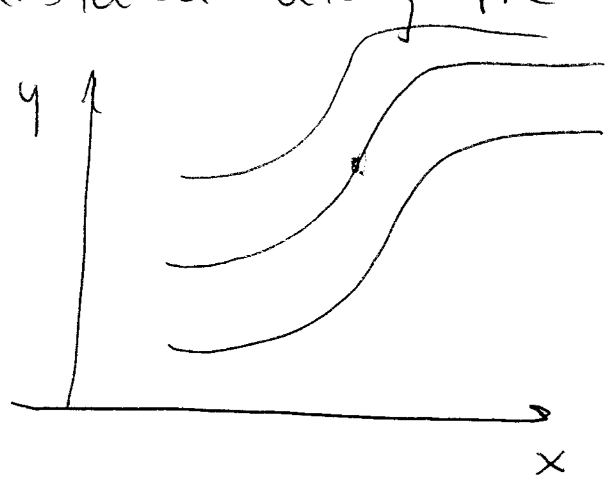
$$y_2'(x) = g(x)$$

$$y_1(x_0) = y_2(x_0) = y_0$$

OR



From before, we know that all solution curves differ by a constant (i.e. are translated along the y-axis)



How do we show uniqueness?

Start by assuming that you have 2 solutions $y_1(x)$ & $y_2(x)$.

Then this means $y_1'(x) = g(x)$ & $y_2'(x) = g(x)$.

Take the difference: $y_1'(x) - y_2'(x)$
$$= g(x) - g(x) = 0$$

i.e. $0 = y_1'(x) - y_2'(x) = (y_1 - y_2)'$

Integrate: $y_1(x) - y_2(x) = C$ for all x.

However, we know that $y_1(x_0) = y_0$ and $y_2(x_0) = y_0$.

Set $x = x_0$, then $C = y_1(x_0) - y_2(x_0)$
$$= y_0 - y_0 = 0$$

So $C = 0$ and $y_1(x) - y_2(x) = 0$ for all x, i.e.
$$\boxed{y_1(x) = y_2(x)} \;!!$$