Math 250A
9/22/08

ex A tree has been transplanted and after x years is growing at a rate of $1 + \frac{1}{(1+x)^2}$ meters per year. After 2 years the tree has reached a height of 5 meters. How tall was the tree when it was transplanted?

translate into a differential eqn.

$$\frac{dy}{dx} = 1 + \frac{1}{(1+x)^2} \longrightarrow y(x) = \int 1 + \frac{1}{(1+x)^2}\, dx$$

make a substitution!

$$u \equiv 1+x$$
$$du = dx$$

$$y(x) = x + \int \frac{1}{(1+x)^2}\, dx$$

$$= x + \int \frac{1}{u^2}\, du$$

$$= x - \frac{1}{u} + C = x - \frac{1}{1+x} + C = y(x)$$

$$y(2) = 5 = 2 - \frac{1}{1+2} + C \longrightarrow C = \frac{10}{3}$$

$$y(0) = 0 - \frac{1}{1+0} + \underset{\frac{10}{3}}{C} \longrightarrow \boxed{y(0) = \frac{7}{3}\ m}$$

don't forget units!

Chain rule: $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

↓ use backwards $\Longleftrightarrow$ integration by substitution

make/find substitution

$\boxed{1}$ if $u = g(x)$, then $\underbrace{\int f[g(x)]\, g'(x)\, dx}_{\text{hard integral to deal with}} = \underbrace{\int f(u)\, du}_{\text{easier integral to deal with}}$

$\boxed{2}$ $\int df[g(x)] = f[g(x)] = \int f'[g(x)]\, g'(x)\, dx$

$\boxed{ex}$ $\int (2x+1)\, e^{x^2+x}\, dx$

$u = x^2 + x$

$du = dx(2x+1)$

$\longrightarrow dx = \frac{1}{2x+1} du$

$= \int (2x+1)\, e^{u} \frac{1}{(2x+1)} du$

$= \int e^{u} du = e^{u} + C = e^{x^2+x} + C$

$\boxed{ex}$ $\int \frac{1}{x \ln x} dx$

$u \equiv \ln x$

$du = \frac{dx}{x} \longrightarrow dx = x\, du$

$= \int \frac{1}{x} u^{-1} x\, du = \int u^{-1} du$

$= \ln|u| + C = \ln|\ln x| + C$

generic forms

a $\int g'(x) e^{g(x)} dx = e^{g(x)} + c$

b $\int \frac{g'(x)}{g(x)} dx = \ln|g(x)| + c$

ex $\int 4x\sqrt{x^2+1}\, dx$

$u = x^2+1$

$du = 2x\,dx \longrightarrow dx = \frac{1}{2x} du$

$= \int 4x\, u^{\frac{1}{2}} \frac{1}{2x}\, du$

$= 2\int u^{\frac{1}{2}} du = 2 \cdot \frac{2}{3} u^{3/2} + c = \frac{4}{3}(x^2+1)^{3/2} + c$

ex $\int \tan x\, dx$

$\tan x = \frac{\sin x}{\cos x}$

$= \int \frac{\sin x}{\cos x} dx$

$u = \cos x$, $du = -\sin x\, dx$

$\longrightarrow dx = -\frac{1}{\sin x} du$

$= \int \frac{\cancel{\sin x}}{u} \cdot -\frac{1}{\sin x} du$

$= -\int \frac{1}{u} du = -\ln|u| + c = -\ln|\cos x| + c$

$-\ln|\cos x| = \ln|\cos x|^{-1} = \ln|\sec x|$

$\boxed{4}$

$\boxed{ex}$ $\int x\sqrt{2x-1}\,dx$

$u = 2x-1$

$du = 2\,dx \rightarrow dx = \frac{1}{2}du$

$= \int x\, u^{\frac{1}{2}} \cdot \frac{1}{2}du$

$\rightarrow x = \frac{1}{2}(u+1)$

$= \frac{1}{4}\int (u+1)u^{\frac{1}{2}}\,du$

$= \frac{1}{4}\int u^{3/2} + u^{\frac{1}{2}}\,du = \frac{1}{4}\left[\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right] + C$

$= \frac{1}{10}(2x-1)^{5/2} + \frac{1}{6}(2x-1)^{3/2} + C$

$\boxed{\text{LIMITS}}$

$\boxed{ex}$ $\int_0^4 2x\sqrt{x^2+1}\,dx$

$u = x^2+1 \rightarrow dx = \frac{1}{2x}du$

$= \int_1^{17} u^{\frac{1}{2}}\,du = \frac{2}{3}u^{3/2}\Big|_1^{17} = \frac{2}{3}(x^2+1)^{3/2}\Big|_0^4$

(*) don't forget to change the limits for definite integrals

$= \frac{2}{3}(17^{3/2}+1) \simeq 47.4$

(exact answer) ∴ (appox.)

$\boxed{ex}$ $\int_1^2 \frac{3x^2+1}{x^3+x}\,dx$

$u = x^3+x,\ du = dx(3x^2+1)$

$\rightarrow dx = \frac{1}{3x^2+1}du$

$= \int_2^{10} u^{-1}\,du$

$= \ln|u|\Big|_2^{10} = \ln 10 - \ln 2$

ideally, you write out all key steps explicitly

$= \ln\left|\frac{10}{2}\right| = \ln 5$

∴