Equations of the form \( y = g(t) \)

1. **Existence**

   (a) \( |t-t_0| \leq a \iff -a \leq t - t_0 \leq a \)

   \( \iff t_0 - a \leq t \leq t_0 + a \)

   Saying that \( |t-t_0| \leq a \) is the same as saying that \( t \in [t_0 - a, t_0 + a] \)

   \[
   \begin{array}{ccc}
   t_0 - a & \rightarrow & t_0 + a
   \end{array}
   \]

   Saying that \( |t-t_0| \leq a \) \& \( |y-y_0| \leq b \)

   is the same as saying that the point with coordinates \((t, y)\) is in the rectangle

   \[
   \begin{array}{c}
   t_0 - a \leq t \leq t_0 + a \\
   y_0 - b \leq y \leq y_0 + b
   \end{array}
   \]
If \( g \) is continuous on the rectangle \( R \), then there will exist at least one solution satisfying \( y(t_0) = y_0 \).

2. **Uniqueness**

Assume \( g(y) = y \)

\[ |g(y_1) - g(y_2)| = |y_1 - y_2| \]

Lipschitz with \( k \geq 1 \).
Assume \( g(y) = y^2 \)

\[
\begin{align*}
|g(y_1) - g(y_2)| &= |y_1^2 - y_2^2| \\
&= |y_1 - y_2| / |y_1 + y_2|.
\end{align*}
\]

\[
|y_1 + y_2| \leq |y_1| + |y_2|
\]

\[
\begin{array}{c}
\text{In } [y_0 - b, y_0 + b], \ |y| \text{ is bounded} \\
\text{say by } a \text{ constant } C
\end{array}
\]

Then,

\[
|g(y_1) - g(y_2)| = \frac{|y_1 - y_2|}{|y_1 + y_2|} \leq \frac{C}{|y_1 - y_2|}.
\]

Recall

\[
\lim_{y_1 \to y_2} \frac{|g(y_1) - g(y_2)|}{|y_1 - y_2|} = |g'(y_2)|
\]
If \( g'(y) \) is continuous on \( \mathbb{R} \), then the solution that goes through \( y(t_0) = y_0 \) is unique.

**Example:** \( g(y) = \sqrt{|y|} \)

\[ \begin{array}{c}
\text{This function is continuous, so there is a solution to } y' = \sqrt{|y|} \text{ such that } y(3) = 0.
\end{array} \]

1. **Is it unique?**
   
   Look at \( \frac{d}{dy} g(y) = \frac{d}{dy} \left( \sqrt{|y|} \right) \)

   For \( y > 0 \) \( g'(y) = \frac{d}{dy} \sqrt{y} = \frac{1}{2\sqrt{y}} = g'(y) \)

   For \( y < 0 \) \( g'(y) = \frac{d}{dy} \sqrt{-y} = g'(y) = \frac{-1}{2\sqrt{-y}} \)
Let us try to solve \( y' = \sqrt{y'} \) \( (y > 0) \)

\[ \frac{dy}{\sqrt{y'}} = dt \Rightarrow 2\sqrt{y'} = t + C \]

\[ \Rightarrow \sqrt{y'} = \frac{t + C}{2} \]

\[ \Rightarrow y = \left( \frac{t + C}{2} \right)^2 = \left( \frac{t}{2} + \frac{C}{2} \right)^2 \]

Set \( y(0) = 0 \) so

\[ 0 = \left( \frac{0}{2} + C \right)^2 \Rightarrow C = -\frac{3}{2} \]

Thus, \( y(t) = \left( \frac{t - 3}{2} \right)^2 \)

However, \( y(t) = 0 \) also satisfies \( y' = \sqrt{y'} \)

and \( y(3) = 0 \).