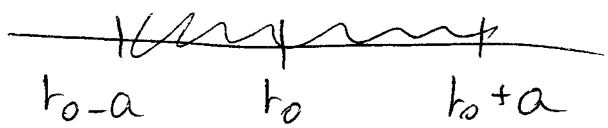
Equations of the form $y' = g(y)$

1. Existence

(a) $|t - t_0| \leq a \iff -a \leq t - t_0 \leq a$

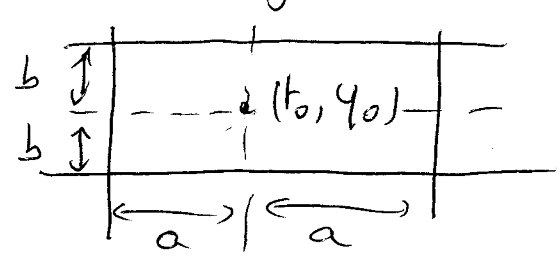
$\iff t_0 - a \leq t \leq t_0 + a$

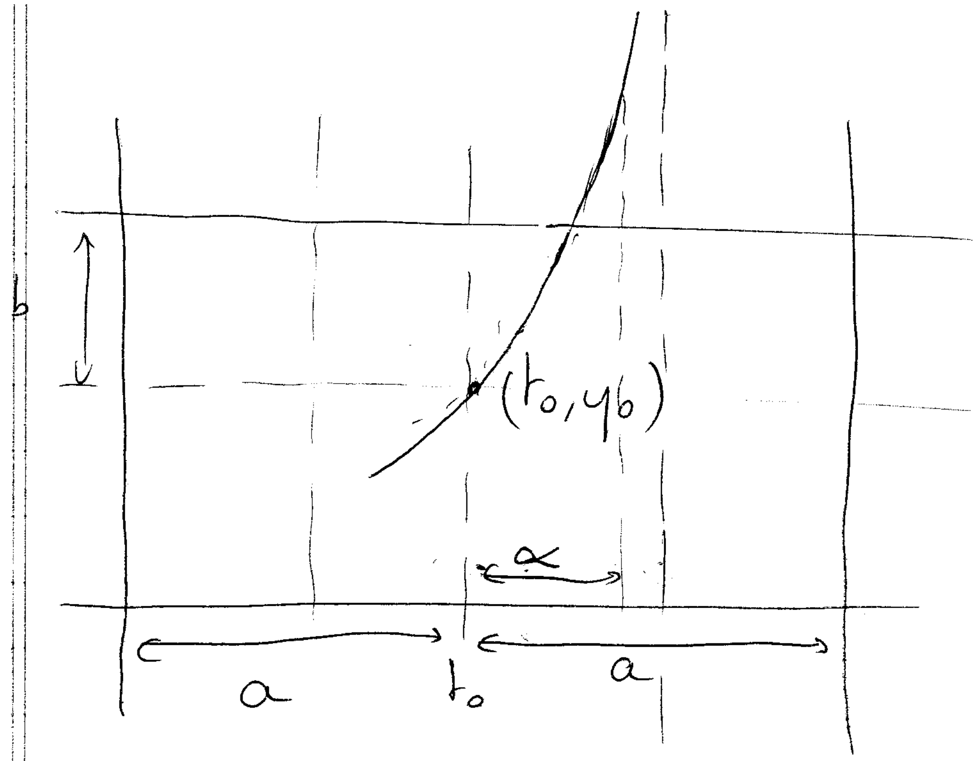
Saying that $|t - t_0| \leq a$ is the same as saying that $t \in [t_0 - a, t_0 + a]$



Saying that $|t - t_0| \leq a$ & $|y - y_0| \leq b$ is the same as saying that the point with coordinates (t, y) is in the rectangle

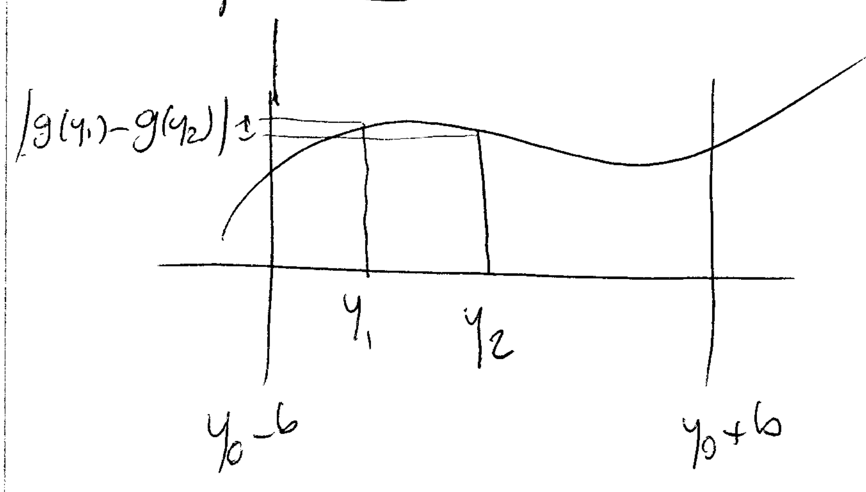
$$\begin{cases} t_0 - a \leq t \leq t_0 + a \\ y_0 - b \leq y \leq y_0 + b \end{cases}$$





If g is continuous on the rectangle R, then there will exist at least one solution satisfying $y(t_0) = y_0$.

2. <u>Uniqueness</u>



<u>Assume $g(y) = y$</u>

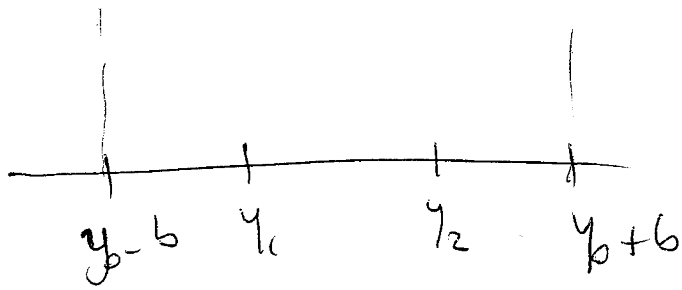
$$|g(y_1) - g(y_2)| \leq |y_1 - y_2|$$

Lipschitz with $k \geq 1$.

<u>Assume $g(y)=y^2$</u>

$$|g(y_1)-g(y_2)| = |y_1^2-y_2^2|$$
$$= |y_1-y_2|\,|y_1+y_2|$$

$$|y_1+y_2| \le |y_1| + |y_2|$$



In $[y_0-b,\ y_0+b]$, $|y|$ is bounded say by a constant C

Then, $|g(y_1)-g(y_2)| = |y_1-y_2|\,|y_1+y_2|$

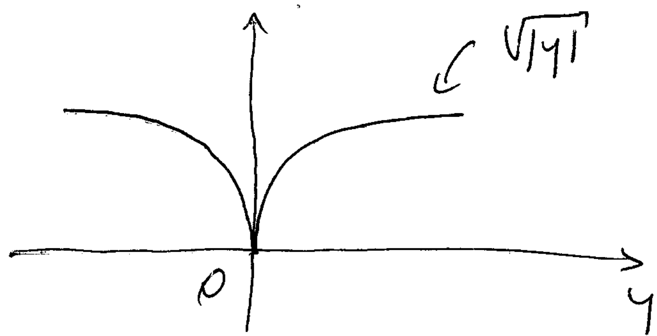
$$\le |y_1-y_2|\, 2C$$

Recall $\dfrac{|g(y_1)-g(y_2)|}{|y_1-y_2|}$

$$\lim_{y_1\to y_2} \frac{|g(y_1)-g(y_2)|}{|y_1-y_2|} = |g'(y_2)|$$

If $g'(y)$ is continuous on $\mathbb{R}$, then the solution that goes through $y(t_0) = y_0$ is unique.

Example: $g(y) = \sqrt{|y|}$



1/ This function is continuous, so there is a solution to $y' = \sqrt{|y|}$ such that $y(3) = 0$.

2/ Is it unique?

Look at $\frac{d}{dy} g(y) = \frac{d}{dy}\left(\sqrt{|y|}\right)$

For $y > 0$ $\quad g'(y) = \frac{d}{dy}\sqrt{y} = \frac{1}{2\sqrt{y}} = g'(y)$

For $y < 0$ $\quad g'(y) = \frac{d}{dy}\sqrt{-y} \quad g'(y) = \frac{-1}{2\sqrt{-y}}$

Let us try to solve $y' = \sqrt{y}$ $(y>0)$

$$\frac{dy}{\sqrt{y}} = dt \Rightarrow 2\sqrt{y} = t + c$$

$$\Rightarrow \sqrt{y} = \frac{t+c}{2}$$

$$\Rightarrow y = \left(\frac{t+c}{2}\right)^2 = \left(\frac{t}{2} + k\right)^2$$

Set $y(3) = 0$ so

$$0 = \left(\frac{3}{2} + k\right)^2 \Rightarrow k = -\frac{3}{2}$$

Thus, $y(t) = \left(\frac{t-3}{2}\right)^2$

However, $y(t) = 0$ also satisfies $y' = \sqrt{y}$

and $y(3) = 0$.