

# Methods of integration II

## 1. Partial fractions

Step 1.

$$\frac{x^3}{x^2+3x+2} = N(x) + \frac{P(x)}{x^2+3x+2}$$

$$\begin{aligned} x^3 &= x(x^2+3x+2) - 3x^2 - 2x \\ &= x(x^2+3x+2) - 3(x^2+3x+2) + 9x + 6 - 2x \\ &= (x-3)(x^2+3x+2) + 7x + 6 \end{aligned}$$

$$\text{So } \frac{x^3}{x^2+3x+2} = x-3 + \frac{7x+6}{x^2+3x+2}$$

$$\text{Here } N(x) = x-3 \quad P(x) = 7x+6$$

$$\frac{x^4}{(x-7)(x+6)} = \frac{x^4}{x^2-x-42} = x^2+x+43 + \frac{85x+42 \cdot 43}{x^2-x-42}$$

$$\begin{aligned} x^4 &= x^2(x^2-x-42) + x^3 + 42x^2 \\ &= x^2(x^2-x-42) + x(x^2-x-42) + x^2 + 42x + 42x^2 \\ &= (x^2+x)(x^2-x-42) + 43(x^2-x-42) + 43x + 42 \cdot 43 \\ &\quad + 42x \\ &= (x^2+x+43)(x^2-x-42) + 85x + 42 \cdot 43 \end{aligned}$$

Step 2.

Partial fraction decomposition

$$f(x) = \frac{x^2 + 5}{(x+1)(x^2 - 2x + 3)} = \frac{P(x)}{Q(x)}$$

$$d^{\circ}P = 2 \quad d^{\circ}Q = 3$$

Factor Q(x)

$$\begin{aligned} (x+1)(x^2 - 2x + 3) &= (x+1)((x-1)^2 - 1 + 3) \\ &= (x+1)((x-1)^2 + 2) \end{aligned}$$

↑  
irreducible

Terms:  $(x+1)$  is like  $(x-a)^n$  with  $\begin{cases} a = -1 \\ n = 1 \end{cases}$

$x^2 - 2x + 3$  is like  $x^2 + bx + c$  with  $\begin{cases} b = -2 \\ c = 3 \\ n = 1 \end{cases}$

Partial fraction decomposition of  $f$ :

$$f(x) = \frac{A}{x+1} + \frac{Bx + C}{x^2 - 2x + 3}$$

Find A:  $(x+1)f(x) = A + \frac{Bx + C}{x^2 - 2x + 3}(x+1)$

set  $x = -1$  and find A

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$$\frac{x^2+5}{(x+1)(x^2-2x+3)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-2x+3}$$

$$\frac{x^2+5}{x^2-2x+3} = A + \frac{Bx+C}{x^2-2x+3} (x+1)$$

$$\text{Set } x = -1 : \frac{6}{1+2+3} = A + 0 \Rightarrow A = 1$$

Find B & C: Multiply by  $x^2-2x+3$

$$\frac{x^2+5}{x+1} = \frac{1}{x+1} (x^2-2x+3) + Bx+C$$

$$\text{Set } x=0 : \frac{5}{1} = \frac{3}{1} + 0 + C \Rightarrow C = 2$$

$$\text{Set } x=1 : \frac{6}{2} = \frac{1-2+3}{2} + B + C$$

$$\Rightarrow 3 = 1 + B + C \Rightarrow B = 2 - C = 0$$

$$\text{So } \frac{x^2+5}{(x+1)(x^2-2x+3)} = \frac{1}{x+1} + \frac{2}{x^2-2x+3}$$

What if we had  $\frac{x^2+5}{(x+1)^3(x^2-2x+3)^2(x-2)^2}$

$$Q(x) = (x+1)^3 ((x-1)^2+2)^2 (x-2)^2$$

Partial fraction decomposition:

$$\begin{aligned} & \frac{x^2+5}{(x+1)^3(x^2-2x+3)^2(x-2)^2} \\ &= \frac{A_3}{(x+1)^3} + \frac{A_2}{(x+1)^2} + \frac{A_1}{x+1} + \frac{B_2x+C_2}{(x^2-2x+3)^2} + \frac{B_1x+C_1}{x^2-2x+3} \\ & \quad + \frac{D_2}{(x-2)^2} + \frac{D_1}{x-2} \end{aligned}$$