Methods of integration II (continued)

1. Partial fractions (continued)

How to integrate $\dfrac{1}{\left(\left(x+\frac{b}{2}\right)^2+d^2\right)^n}$

Re-write $\dfrac{1}{\left(\left(x+\frac{b}{2}\right)^2+d^2\right)^n} = \dfrac{1}{\left[d^2\left(\left(\frac{x+b/2}{d}\right)^2+1\right)\right]^n}$

$$= \frac{1}{d^{2n}\left(\left(\frac{x+b/2}{d}\right)^2+1\right)}$$

Let $u = \dfrac{x}{d} + \dfrac{b}{2d}$ $\quad du = \dfrac{dx}{d}$

$$\int \frac{1}{\left[\left(x+\frac{b}{2}\right)^2+d^2\right]^n} = \frac{1}{d^{2n}} \int \frac{dx}{\left(\left(\frac{x+b/2}{d}\right)^2+1\right)^n}$$

$$= \frac{1}{d^{2n}} \int \frac{d\,du}{(1+u^2)^n} = \frac{1}{d^{2n-1}} \int \frac{du}{(1+u^2)^n}$$

If $n=1$ $\quad \displaystyle\int \frac{du}{1+u^2} = \arctan(u) + C$

If $n>1$ $\quad$ Let $u = \tan(\theta)$

$$\frac{d}{d\theta}\tan(\theta) = \frac{1}{\cos^2(\theta)} = \sec^2(\theta) = 1+\tan^2(\theta)$$

$$du = d\theta \frac{du}{d\theta} = d\theta \left(1+\tan^2(\theta)\right)$$
$$= d\theta\,(1+u^2)$$

$$\int \frac{du}{(1+u^2)^n} = \int \frac{d\theta\,(1+u^2)}{(1+u^2)^n}$$
$$= \int \frac{d\theta}{(1+u^2)^{n-1}} \qquad n>1$$

$$1+u^2 = 1+\tan^2\theta = \frac{1}{\cos^2\theta}$$

$$\int \frac{du}{(1+u^2)^n} = \int \frac{d\theta}{(1+u^2)^{n-1}} = \int (\cos^2\theta)^{n-1}\,d\theta$$
$$= \int \cos^{2n-2}(\theta)\,d\theta$$

<u>Other method</u>: To find $\int \frac{du}{(1+u^2)^n}$ $n>1$

Method: integrate $\frac{1}{(1+u^2)^{n-1}}$ by parts

<u>Example</u>: Find $\frac{1}{(1+u^2)^2}$

Start $\int \frac{du}{1+u^2}$ & integrate by parts

Let $v = \dfrac{1}{1+u^2}$ $\qquad dq = du$

$$dv = \frac{-2u}{(1+u^2)^2}\, du$$

$$q = u$$

$$\int \frac{du}{1+u^2} = \frac{u}{1+u^2} - \int u \frac{-2u}{(1+u^2)^2}\, du$$

$$= \frac{u}{1+u^2} + 2\int \frac{u^2}{(1+u^2)^2}\, du$$

$$= \frac{u}{1+u^2} + 2\int \frac{(u^2+1)-1}{(1+u^2)^2}\, du$$

$$= \frac{u}{1+u^2} + 2\int \frac{du}{1+u^2} - 2\int \frac{du}{(1+u^2)^2}$$

$$2\int \frac{du}{(1+u^2)^2} = \frac{u}{1+u^2} + \int \frac{du}{1+u^2}$$

$$\Rightarrow \int \frac{du}{(1+u^2)^2} = \frac{1}{2}\,\frac{u}{1+u^2} + \frac{1}{2}\int \frac{du}{1+u^2}$$

$$= \frac{1}{2}\,\frac{u}{1+u^2} + \frac{1}{2}\arctan(u) + C$$

Examples of application

1. Solve $\dfrac{dy}{dx} = \dfrac{(y+1)(y^2-2y+3)}{y^2+5}$

$$\frac{y^2+5}{(y+1)(y^2-2y+3)}\, dy = dx \qquad y \neq -1$$

$$y^2 - 2y + 3 = (y-1)^2 + 2$$

Then $\quad x + C = \displaystyle\int \frac{y^2+5}{(y+1)(y^2-2y+3)}\, dy$

$$= \int \left(\frac{1}{y+1} + \frac{2}{y^2-2y+3} \right) dy$$

$$= \ln|y+1| + 2 \int \frac{dy}{(y-1)^2+2}$$

$$= \ln|y+1| + 2 \int \frac{dy}{2\left[1 + \left(\frac{y-1}{\sqrt{2}}\right)^2\right]}$$

$$= \ln|y+1| + \int \frac{dy}{1 + \left(\frac{y-1}{\sqrt{2}}\right)^2}$$

$$x + C = \ln|y+1| + \sqrt{2} \arctan\left(\frac{y-1}{\sqrt{2}}\right)$$

Example 2: $\frac{dy}{dx} = \frac{(y-1)^3}{y^4}$

with initial conditions (a) $y(0) = 2$

(b) $y(0) = 1$

$$\frac{y^4}{(y-1)^3}\, dy = dx$$

$$x + C = \int \frac{y^4}{(y-1)^3}\, dy$$

1. Long division $\qquad (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$$\frac{y^4}{(y-1)^3} = \frac{y^4}{y^3 - 3y^2 + 3y - 1}$$

$$= \frac{y(y^3 - 3y^2 + 3y - 1) + 3y^3 - 3y^2 + y}{y^3 - 3y^2 + 3y - 1}$$

$$= y + \frac{3(y^3 - 3y^2 + 3y - 1) + 6y^2 - 8y + 3}{y^3 - 3y^2 + 3y - 1}$$

$$= y + 3 + \frac{6y^2 - 8y + 3}{(y-1)^3}$$

$$= y + 3 + \frac{A}{y-1} + \frac{B}{(y-1)^2} + \frac{C}{(y-1)^3}$$

Find C: multiply across by $(y-1)^3$ & set $y = 1$

$$\frac{y^4}{(y-1)^3} = y+3+\frac{6y^2-8y+3}{(y-1)^3}$$

$$\frac{6y^2-8y+3}{(y-1)^3} = \frac{A}{y-1}+\frac{B}{(y-1)^2}+\frac{C}{(y-1)^3}$$

$$\Rightarrow \quad 6y^2-8y+3 = A(y-1)^2+B(y-1)+C$$

Set $y=1$ $\quad C = 6-8+3 = 1$

$y=2$ $\quad 24-16+3 = A+B+C$

$\Rightarrow A+B = 11-C = 11-1 = 10$

$y=0$ $\quad 3 = A-B+C \Rightarrow A-B = 3-C = 2$

So $\begin{cases} A+B = 10 \\ A-B = 2 \end{cases} \Rightarrow \begin{cases} 2A = 12 \\ 2B = 8 \end{cases} \Rightarrow \begin{cases} A = 6 \\ B = 4 \end{cases}$

So $$\frac{6y^2-8y+3}{(y-1)^3} = \frac{6}{y-1}+\frac{4}{(y-1)^2}+\frac{1}{(y-1)^3}$$

and $$\frac{y^4}{(y-1)^3} = y+3+\frac{6}{y-1}+\frac{4}{(y-1)^2}+\frac{1}{(y-1)^3}$$

Therefore $$x+C = \int \frac{y^4}{(y-1)^3}\,dy$$

i.e. $$x+C = \frac{1}{2}y^2+3y+6\ln|y-1|-\frac{4}{y-1}+\frac{-1/2}{(y-1)^2}$$