Numerical approximations (continued)

3. Taylor Polynomials (continued)

Example: $f(x) = \tan(x)$

Find Taylor polynomials near $x = 0$
i.e. $a = 0$

$$P_0(x) = \tan(0) = 0$$

$$P_1(x) = P_0(x) + \frac{d}{dx}\tan(x)\Big|_{x=0} x$$

$$\frac{d}{dx}\tan(x) = \frac{1}{\cos^2(x)} \Rightarrow \tan'(0) = 1$$

So $P_1(x) = 0 + 1 \cdot x = x$

$$P_2(x) = P_1(x) + \tan''(0)\frac{x^2}{2}$$

$$\frac{d^2}{dx^2}\tan(x) = \frac{d}{dx}\left(1 + \tan^2(x)\right) = 2\tan(x)\left(1 + \tan^2(x)\right)$$

$$\tan''(0) = 0$$

So $P_2(x) = P_1(x)$

$$P_3(x) = P_2(x) + \tan'''(0)\frac{x^3}{3!}$$

$$\frac{d^3}{dx^3}\tan(x) = \frac{d}{dx}\left(2\tan(x)\left(1 + \tan^2(x)\right)\right)$$
$$= 2\left(1 + \tan^2(x)\right)^2 + 2\tan^2(x) \cdot 2\left(1 + \tan^2(x)\right)$$

So $$\frac{d^3}{dx^3}\tan(x)\Big|_{x=0} = 2$$

and $$P_3(x) = P_2(x) + 2\,\frac{x^3}{6}$$
$$= x + \frac{x^3}{3}$$

<u>Error</u> :

For x near a, we can write

$$\tan(x) = P_3(x) + \frac{(x-\overset{a=0}{a})^4}{24}\tan^{(4)}(\xi) = x + \frac{x^3}{3} + \frac{x^4}{24}\tan^{(4)}(\xi)$$

where ξ is between a & x.

This is in fact true for all x's, as long as f is smooth between a & x.

<u>Note</u> :

$$f(x) = f(a) + \int_a^x f'(t)\,dt$$

We can find the Taylor polynomials of f near $x = a$ by integration by parts.

This will also give us the remainder $R_n(x)$ in the form of an integral.

$$f(x) = f(a) + \int_a^x f'(t)\,dt$$

$$= f(a) + \Big[(t-x)\,f'(t)\Big]_a^x - \int_a^x f''(t)\,(t-x)\,dt$$

$$= f(a) - (a-x)\,f'(a) - \int_a^x f''(t)\,(t-x)\,dt$$

$$= f(a) + (x-a)\,f'(a) - \left[\frac{(t-x)^2}{2} f''(t)\right]_a^x + \int_a^x \frac{(t-x)^2}{2} f'''(t)\,dt$$

$$= f(a) + (x-a)\,f'(a) + \frac{(x-a)^2}{2} f''(a) + \left[\frac{(t-x)^3}{6} f'''(t)\right]_a^x - \int_a^x \frac{(t-x)^3}{6} f^{(4)}(t)\,dt$$

$$= f(a) + (x-a)\,f'(a) + \frac{(x-a)^2}{2} f''(a) + \frac{(x-a)^3}{3!} f'''(a) - \int_a^x \frac{(t-x)^3}{6} f^{(4)}(t)\,dt$$

4. Approximation of integrals (revisited)

We want to figure out how much of an error we make when we use say LEFT(1) to evaluate $\int_{x_i}^{x_{i+1}} f(x)\,dx$ where $x_{i+1} - x_i = \Delta x$.

Write $f(x) = f(x_i) + (x - x_i)\, f'(\xi(x)) \qquad \xi(x) \in (x_i, x)$

Note that $\displaystyle\int_{x_i}^{x_{i+1}} f(x)\,dx$ is what we are looking for.

Moreover, $\displaystyle\int_{x_i}^{x_{i+1}} f(x_i)\,dx = f(x_i)(x_{i+1} - x_i) = f(x_i)\Delta x$

$$= LEFT(1)$$

Now, I can write

$$f(x) - f(x_i) = (x - x_i)\, f'(\xi(x)).$$

Assume f is increasing on $[a,b]$, $f'(x)$ is positive on $[a,b]$, so $f'(\xi(x))$ is also positive.
Between x_i & x_{i+1}, $x - x_i \geq 0$, so that

$$f(x) - f(x_i) = (x - x_i)\, f'(\xi(x)) \geq 0$$

$$\Rightarrow f(x) \geq f(x_i)$$

$$\Rightarrow \int_{x_i}^{x_{i+1}} f(x)\,dx \geq \int_{x_i}^{x_{i+1}} f(x_i)\,dx$$

i.e. $\displaystyle\int_{x_i}^{x_{i+1}} f(x)\,dx \geq LEFT(1)$

Also, from $f(x) - f(x_i) = (x - x_i)\, f'(\xi(x))$

we have $|f(x) - f(x_i)| = |x - x_i|\,|f'(\xi(x))|$

Now since $f(x) - f(x_i) \geq 0$, $|f(x) - f(x_i)| = f(x) - f(x_i)$

So I can write

$$f(x) - f(x_i) = [x - x_i] \, |f'(\xi(x))|$$

$$\leq (x - x_i) \, M$$ where M is such that $|f'(x)| \leq M$ on $[a, b]$

$$\Rightarrow \int_{x_i}^{x_{i+1}} f(x)\,dx - \int_{x_i}^{x_{i+1}} f(x_i)\,dx \leq \int_{x_i}^{x_{i+1}} (x - x_i) \, M \, dx$$

$$\Rightarrow \int_{x_i}^{x_{i+1}} f(x)\,dx - \text{LEFT}(1) \leq M \left[\frac{(x - x_i)^2}{2} \right]_{x_i}^{x_{i+1}}$$

$$= M \frac{(x_{i+1} - x_i)^2}{2}$$

$$= M \frac{\Delta x^2}{2}$$

As a consequence, the error is bounded by $\frac{\Delta x^2}{2} M$.

If the error on each subinterval is of order $\Delta x^2 = \frac{(b-a)^2}{n^2}$, then the error on $\int_a^b f(x)\,dx$ is of order $n \, \Delta x^2$ i.e. it goes like $\frac{1}{n}$.