2. Taylor Polynomials (continued)

Example: \( f(x) = \tan(x) \)

Find Taylor polynomials near \( x = 0 \), i.e., \( a = 0 \)

\[ P_0(x) = \tan(0) = 0 \]

\[ P_1(x) = P_0(x) + \frac{d}{dx} \tan(x) \bigg|_{x=0} \]

\[ \frac{d}{dx} \tan(x) = \frac{1}{\cos^2(x)} \Rightarrow \tan'(0) = 1 \]

So \( P_1(x) = 0 + 1 \cdot x = x \)

\[ P_2(x) = P_1(x) + \tan''(0) \frac{x^2}{2} \]

\[ \frac{d^2}{dx^2} \tan(x) = \frac{d}{dx} \left( 1 + \tan^2(x) \right) = 2 \tan(x) \left( 1 + \tan^2(x) \right) \]

\[ \tan''(0) = 0 \]

So \( P_2(x) = P_1(x) \)

\[ P_3(x) = P_2(x) + \tan'''(0) \frac{x^3}{3!} \]

\[ \frac{d^3}{dx^3} \tan(x) = \frac{d}{dx^2} \left( 2 \tan(x) \left( 1 + \tan^2(x) \right) \right) = \frac{d}{dx} 2 \left( 1 + \tan^2(x) \right)^2 + 2 \tan(x) \cdot 2 \left( 1 + \tan^2(x) \right) \]
So \( \frac{d^3}{dx^3} \tan(x) \bigg|_{x=0} = 2 \)

and \( P_3(x) = P_2(x) + 2 \frac{x^3}{6} \)

\[ = x + \frac{x^3}{3} \]

Error:

For \( x \) near \( a \), we can write

\[ \tan(x) = P_3(x) + \frac{(x-a)^4}{24} \tan^{(4)}(\xi) = x + \frac{x^3}{3} + \frac{x^4}{24} \tan^{(4)}(\xi) \]

where \( \xi \) is between \( a \) & \( x \).

This is in fact true for all \( x \)'s, as long as \( f \) is smooth between \( a \) & \( x \).

Note:

\[ f(x) = f(a) + \int_a^x f'(t) \, dt \]

We can find the Taylor polynomials of \( f \) near \( x = a \) by integration by parts.

This will also give us the remainder \( R_n(x) \) in the form of an integral.
\[ f(x) = f(a) + \int_a^x f'(t) \, dt + \int_a^x f''(t) (t-x) \, dt \]
\[ = f(a) + \left[ (t-x) f'(t) \right]_a^x - \int_a^x f''(t) (1-x) \, dt \]
\[ = f(a) - (x-a) f'(a) - \int_a^x f''(t) (1-x) \, dt \]
\[ = f(a) + (x-a) f'(a) - \left[ \frac{(t-x)^2}{2} f''(t) \right]_a^x \]
\[ + \int_a^x \frac{(t-x)^2}{2} f'''(t) \, dt \]
\[ = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2} f''(a) \]
\[ + \left[ \frac{(t-x)^3}{6} f'''(t) \right]_a^x - \int_a^x \frac{(t-x)^3}{6} f^{(4)}(t) \, dt \]
\[ = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2} f''(a) \]
\[ + \frac{(x-a)^3}{3!} f'''(a) - \int_a^x \frac{(t-x)^3}{6} f^{(4)}(t) \, dt \]

9. Approximation of integrals (revisited)

We want to figure out how much of an error we make when we use say \text{LEFT}(1)

to evaluate \[ \int_{x_i}^{x_{i+1}} f(x) \, dx \] where \( x_{i+1} - x_i = Ax \).
Write \( f(x) = \int \frac{f(x_i)}{(x-x_i) f'(\xi(x))} \) for \( \xi(x) \in (x_i, x) \).

Note that \( \int_{x_i}^{x_{i+1}} \frac{f(x)}{x} \, dx \) is what we are looking for.

Moreover, \( \int_{x_i}^{x_{i+1}} \frac{f(x_i)}{(x-x_i+1)} \, dx = f(x_i) (x_{i+1} - x_i) = f(x_i) \Delta x \)

\( = \text{LEFT}(1) \)

Now, I can write \( f(x) - f(x_i) = (x-x_i) f'(\xi(x)) \).

Assume \( f \) is increasing on \((a,b)\), \( f'(x) \) is positive on \([a,b]\), so \( f'(\xi(x)) \) is also positive.

Between \( x_i \) and \( x_{i+1} \), \( x-x_i \geq 0 \), so that

\( f(x) - f(x_i) = (x-x_i) f'(\xi(x)) \geq 0 \)

\( \Rightarrow f(x) \geq f(x_i) \)

\( = \int_{x_i}^{x_{i+1}} f(x) \, dx \geq \int_{x_i}^{x_{i+1}} f(x_i) \, dx \)

i.e. \( \int_{x_i}^{x_{i+1}} f(x) \, dx \geq \text{LEFT}(1) \)

Also, from \( f(x) - f(x_i) = (x-x_i) f'(\xi(x)) \)

we have \( |f(x) - f(x_i)| = |x-x_i| |f'(\xi(x))| \)
Now since \( f(x) - f(x_i) \geq 0 \), \( |f(x) - f(x_i)| = \frac{f(x) - f(x_i)}{f(x) - f(x_i)} \)

So I can write

\[
f(x) - f(x_i) = \int_{x_i}^{x} f'(\xi(x)) \, d\xi \leq (x - x_i) \, M \text{ where } M \text{ is such that } |f'(x)| \leq M \text{ on } [a, b]
\]

\[
\Rightarrow \int_{x_i}^{x_{i+1}} f(x) \, dx - \int_{x_i}^{x_i} f(x) \, dx \leq \int_{x_i}^{x_{i+1}} (x - x_i) \, M \, dx
\]

\[
\Rightarrow \int_{x_i}^{x_{i+1}} f(x) \, dx - \text{LEFT}(1) \leq M \left[ \frac{(x - x_i)^2}{2} \right]_{x_i}^{x_{i+1}} = M \frac{(x_{i+1} - x_i)^2}{2} = M \frac{\Delta x^2}{2}
\]

As a consequence, the error is bounded by \( \frac{\Delta x^2 \, M}{2} \).

If the error on each subinterval is of order \( \Delta x^2 = \frac{(b-a)^2}{n^2} \), then the error on \( \int_{a}^{b} f(x) \, dx \) is of order \( n \Delta x^2 \) i.e. it goes like \( \frac{1}{n} \).