More about the $\xi$

Recall that we had

$$f(x) = P_3(x) - \int_a^x \frac{(t-x)^3}{6} f^{(4)}(t) \, dt$$

We want to re-write this

$$f(x) = P_3(x) + \frac{(x-a)^4}{4!} f^{(4)}(\xi)$$

$$= P_3(x) - \int_a^x \frac{(t-x)^3}{6} f^{(4)}(t) \, dt$$

So we want to find $\xi$ such that

$$- \int_a^x \frac{(t-x)^3}{6} f^{(4)}(t) \, dt = \frac{(x-a)^4}{4!} f^{(4)}(\xi)$$

i.e.,

$$f^{(4)}(\xi) = -\frac{4!}{(x-a)^4} \int_a^x \frac{(t-x)^3}{6} f^{(4)}(t) \, dt = G(x)$$

Here, $\xi$ is known i.e. $G(x)$ is just a number, that depends on $x$. 
Instability of Euler's method

\[ y' = \frac{dy}{dx} = \frac{dy}{dx} = \frac{dy}{dx} = dx \frac{dy}{dx} \]

i.e. \( \ln |y| = dx + c \)

i.e. \( y = e^c e^{dx} = k e^{dx} \)

Exact solution is \( y = k e^{dx} \)

If the initial condition is \( y(0) = y_0 \), then \( y = y_0 e^{dx} \).

Assume that \( d < 0 \). \( \lim_{x \to -\infty} y(x) = 0 \)

Apply Euler's method to this:

\[ y_{n+1} = y_n + h g(x_n, y_n) \quad y' = g(x, y) \quad \frac{y_{n+1} - y_n}{dx} = g(x_n, y_n) \]

i.e. \( y_{n+1} = y_n + h dy = (1 + dh) y_n \)

Say for \( n = 0 \) \( y = y_0 \)

\( n = 1 \) \( y_1 = (1 + Ah) y_0 \)

\( n = 2 \) \( y_2 = (1 + Ah)^2 y_0 \)

\( n \) \( y_n = (1 + Ah)^n y_0 \)
so if \( |1 + dh| < 1 \)

then \( \lim_{n \to \infty} y_n = \lim_{n \to \infty} (1 + dh)^n y_0 = 0 \)

However, if \( |1 + dh| > 1 \)

then \( \lim_{n \to \infty} |y_n| = \lim_{n \to \infty} |1 + dh|^n |y_0| = +\infty \)

If \( 1 + dh < 0 \), then the sign of \( y_n \) alternates

If \( |1 + dh| = 1 \), then \( |y_n| = |(1 + dh)^n| |y_0| = |y_0| \)

Say \( 1 + dh = -1 \). Then, \( y_n = (-1)^n y_0 \)
Another potential problem

Nonlinear maps can be chaotic.

For a numerical method, say we have

\[ y_{n+1} = F(y_n, x_n, h) \]

Example of the logistic map

\[ y_{n+1} = a \cdot y_n \cdot (1 - y_n) \]

Colwes diagram