11/24/08

$\boxed{\text{Equations of the form } y' = g(x,y)}$

1. Separable equations

Example: $x^2 y' = y(y-1)$

$$x^2 \frac{dy}{dx} = y(y-1)$$

$$\frac{dy}{y(y-1)} = \frac{dx}{x^2} \qquad y \neq 0, 1$$

Integrate both sides

$$\int \frac{dy}{y(y-1)} = \int \frac{dx}{x^2}$$

$$\Rightarrow \int \left(\frac{-1}{y} + \frac{1}{y-1}\right) dy = \frac{-1}{x} + c$$

$$\Rightarrow -\ln|y| + \ln|y-1| = \frac{-1}{x} + c$$

$$\Rightarrow \ln\left|\frac{y-1}{y}\right| = -\frac{1}{x} + c$$

$$\Rightarrow \frac{y-1}{y} = \pm e^{-\frac{1}{x}+c} = \pm e^{c} e^{-1/x} = k e^{-1/x}$$

$x \neq 0$

Solve for y:

$$y-1 = \mathcal{K} y e^{-1/x} \Rightarrow y\left(1-\mathcal{K} e^{-1/x}\right) = 1$$

$$\Rightarrow y = \frac{1}{1-\mathcal{K} e^{-1/x}} \qquad \text{recall } \mathcal{K} \neq 0$$

$\mathcal{K}=0$ gives $y=1$ so we can write the general solution as

$$y = \frac{1}{1-\mathcal{K} e^{-1/x}} \qquad \mathcal{K} \text{ arbitrary}$$

There is then 1 singular solution, $y=0$, which can be "recovered" by sending $\mathcal{K}$ to $-\infty$.

Check your answer:

$$x^2 y' = y(y-1) \quad ; \quad y = \frac{1}{1-\mathcal{K} e^{-1/x}}$$

$$y' = \frac{-1(-\mathcal{K}) e^{-1/x} \frac{1}{x^2}}{\left(1-\mathcal{K} e^{-1/x}\right)^2} = \frac{\mathcal{K} e^{-1/x}}{x^2\left(1-\mathcal{K} e^{-1/x}\right)^2}$$

$$= \frac{\mathcal{K} e^{-1/x}}{x^2} y^2 = \frac{\left(1-\frac{1}{y}\right) y^2}{x^2} = \frac{y(y-1)}{x^2} \quad \checkmark$$

$$1-\mathcal{K} e^{-1/x} = \frac{1}{y} \Rightarrow \mathcal{K} e^{-1/x} = 1-\frac{1}{y}$$

2. Equations with homogeneous coefficients

$$y' = g\left(\frac{y}{x}\right)$$

Let $z = \dfrac{y}{x} \Rightarrow y = xz$

Then $y' = \dfrac{dy}{dx} = \dfrac{d}{dx}(xz)$

$$= z + x\frac{dz}{dx} = z + xz'$$

With $y' = g\left(\frac{y}{x}\right)$, we have

$$z + xz' = y' = g\left(\frac{y}{x}\right) = g(z)$$

i.e. $z' = \dfrac{g(z) - z}{x}$ separable

$\hookrightarrow$ Find z as a function of x

$\hookrightarrow$ Then write $y = xz$.

Example: $y' = \dfrac{x^2 + xy + y^2}{xy}$

To see if this is homogeneous, set $y = \alpha x$ where α is a number.

$$\frac{x^2 + xy + y^2}{xy} = \frac{x^2 + x\alpha x + \alpha^2 x^2}{x\alpha x} = \frac{1 + \alpha + \alpha^2}{\alpha}$$

If $g(x, \alpha x)$ does not depend on x, then g is homogeneous of degree 0.

↳ we can apply the method of homogeneous coefficients.

$$y' = \frac{x^2 + xy + y^2}{xy} = \frac{x}{y} + 1 + \frac{y}{x}$$

Let $z = \frac{y}{x}$ $\quad y = zx$

$$y' = z'x + \underline{z} = \left(\frac{x}{y} + 1 + \frac{y}{x}\right) = \frac{1}{z} + 1 + \underline{z}$$

$\Rightarrow z' = \frac{1}{x}\left(\frac{1}{z} + 1\right)$ is separable.