Equations of the form $y' = g(x, y)$ (continued)

Example : $y' = \dfrac{x^2 + xy + y^2}{xy} = \dfrac{x}{y} + 1 + \dfrac{y}{x}$

Let $z = \dfrac{y}{x}$, then $y = xz$ so

$$y' = z + xz' = \frac{1}{z} + 1 + z$$

$$\Rightarrow \quad xz' = \frac{1}{z} + 1 = \frac{1+z}{z}$$

$$\Rightarrow \quad \frac{dz}{dx} = x^{-1} \frac{1+z}{z}$$

i.e. $\dfrac{z}{1+z} \, dz = \dfrac{dx}{x}$

So $\displaystyle \int \frac{dx}{x} = \int \frac{z}{1+z} \, dz = \int \frac{1+z-1}{1+z} \, dz$

$$= \int \left(1 - \frac{1}{1+z}\right) dz$$

i.e.

$$\ln|x| + c = z - \ln|1+z|$$

i.e. $\ln|x(1+z)| = z - c$

i.e. $x(1+z) = \pm e^z e^{-c} = K e^z$

With $z = \dfrac{y}{x}$, we have

$$x\left(1+\frac{y}{x}\right) = k\, e^{y/x}$$

i.e $\boxed{x + y = k\, e^{y/x}}$,

<u>Check</u>: Differentiate both sides with respect to x:

$$1 + y' = k\, e^{y/x}\left(\frac{y'}{x} - \frac{y}{x^2}\right)$$

i.e. $1 + y' = (x+y)\left(\frac{y'}{x} - \frac{y}{x^2}\right)$

$$\Rightarrow\ y'\left(1 - \frac{x+y}{x}\right) = -1 - \frac{y}{x^2}(x+y)$$

$$\Rightarrow\ y'\ \frac{-y}{x} = -\ \frac{x^2 + y(x+y)}{x^2}$$

$$\Rightarrow\ y' = \frac{x}{y}\ \frac{x^2+xy+y^2}{x^2} = \frac{x^2+xy+y^2}{xy}\ \checkmark$$

3. <u>Linear equations</u>

$$y' + P(x)\, y = q(x)$$

$$y' f(x) + y\, g(x) = \frac{d}{dx}\left(y f \right)$$
$$= y' f + y \underbrace{f'}_{g}$$

$$h(x)\left(y' + p(x)\, y\right) = \frac{d}{dx}\left(y\, h(x) \right)$$

$$p(x)\, h(x) = h'(x) = \frac{dh}{dx}$$

$$\Rightarrow \quad \frac{dh}{h} = p(x)\, dx$$

$$\ln|h| = \int p(x)\, dx + C$$

$$h(x) = \pm e^{C} \exp\left(\int p(x)\, dx \right)$$

Pick $h(x) = \exp\left(\int p(x)\, dx \right)$

Then $\exp\left(\int p(x)\, dx \right) \left(y' + p(x)\, y \right)$
$$= \frac{d}{dx}\left(y \exp\left(\int p(x)\, dx \right) \right)$$

Go back to linear equation

$$y' + p(x)\, y = q(x)$$

Multiply by $h(x) = \exp\left(\int p(x)\,dx\right)$

Then

$$\exp\left(\int p(x)\,dx\right)\left(y' + p(x) y\right)$$
$$= q(x) \exp\left(\int p(x)\,dx\right)$$

Realize that the left-hand-side is an exact derivative:

$$\exp\left(\int p(x)\,dx\right)\left(y' + p(x)\, y\right)$$
$$= \frac{d}{dx}\left(\exp\left(\int p(x)\,dx\right) y\right)$$

So the ode reads

$$\frac{d}{dx}\left(\exp\left(\int p(x)\,dx\right) y\right) = q(x) \exp\left(\int p(x)\,dx\right)$$

Integrate:

$$\exp\left(\int p(x)\,dx\right) y = \int q(x) \exp\left(\int p(x)\,dx\right) dx + C$$

Solve for y:

$$y = \frac{\int q(x) \exp\left(\int p(x)\,dx\right) dx + C}{\exp\left(\int p(x)\,dx\right)}$$

$$= \exp\left(-\int p(x)\,dx\right) \int q(x) \exp\left(\int p(x)\,dx\right) dx$$

$$+ C \exp\left(-\int p(x)\,dx\right)$$

<u>What to know</u>: if the equation is first order & linear,

$$y' + p(x)\,y = q(x)$$

remember that multiplying the ode by the exponential of the antiderivative of p makes the left-hand-side an exact derivative. Then solve.

<u>Example</u>: RL circuit

$$L \frac{dI}{dt} + RI = E(t)$$

Linear, first order ode.

$$\frac{dI}{dt} + \frac{R}{L} I = \frac{E(t)}{L}$$

Multiply by $\exp\left(\int p(t)\,dt\right) = \exp\left(\frac{R}{L}t\right)$

then:

$$\exp\left(\frac{R}{L}t\right)\frac{dI}{dt} + \frac{R}{L}\exp\left(\frac{R}{L}t\right)I = \frac{E(t)}{L}\exp\left(\frac{R}{L}t\right)$$

i.e.

$$\frac{d}{dt}\left(I e^{Rt/L}\right) = \frac{E(t)}{L} e^{Rt/L}$$

Integrate:

$$I e^{Rt/L} = \int_0^t \frac{E(s)}{L} e^{Rs/L}\,ds + C$$

i.e. $$I(t) = C e^{-Rt/L} + e^{-Rt/L}\int_0^t \frac{E(s)}{L} e^{Rs/L}\,ds$$

If E is constant, $E(t) = E_0$

Then $$I(t) = C e^{-Rt/L} + e^{-Rt/L}\int_0^t \frac{E_0}{L} e^{Rs/L}\,ds$$

$$= C e^{-Rt/L} + \frac{E_0}{L} e^{-Rt/L}\left[\frac{L}{R} e^{Rs/L}\right]_0^t$$

$$= C e^{-Rt/L} + \frac{E_0}{R} e^{-Rt/L}\left(e^{Rt/L} - 1\right)$$

$$= \tilde{C} e^{-Rt/L} + \frac{E_0}{R}$$