Equations of the form $y'=g(x,y)$ (continued)

3. Linear equations (continued)

Example: $xy'+y = \cos(x)$

You can see that the left-hand-side is $(xy)'$ so that the ode is

$(xy)' = \cos(x)$

i.e. $xy = \int \cos(x) \, dx + C$

$= \sin(x) + C$

i.e. $y(x) = \frac{1}{x} \left( \sin(x) + C \right)$

Apply the method of solution for linear equations (if you don't see that the l.h.s. is an exact derivative):

Re-write as $y' + \frac{1}{x}y = \frac{\cos(x)}{x}$

Method says: multiply by $\exp\left(\int \frac{dx}{x}\right)$

i.e. $\exp(\ln|x|) = |x| \leftarrow$ use $x$ instead,
Example 2: \( y' + \ln(x) y = \tan(x) \)

Multiply by \( \exp\left(\int \ln(x) \, dx\right) \)

\[
\int \ln(x) \, dx = x \ln(x) - \int x \frac{1}{x} \, dx
\]

\[= x \ln(x) - x \left( + C \right) \]

Multiply by \( \exp\left( x \ln(x) - x \right) \)

Then

\[
y' \exp\left( x \ln(x) - x \right) + \frac{\ln(x)}{x} y \exp\left( x \ln(x) - x \right) \]

\[= \tan(x) \exp\left( x \ln(x) - x \right) \]

i.e.

\[
\frac{d}{dx} \left[ y \exp\left( x \ln(x) - x \right) \right] = \tan(x) \exp(x \ln(x) - x) \]

i.e.

\[
y \exp\left( x \ln(x) - x \right) \]

\[= \int \tan(x) \exp\left( x \ln(x) - x \right) \, dx + C \]

i.e.

\[
y(x) = \exp\left( -x \ln(x) + x \right) \int \tan(x) \exp\left( x \ln(x) - x \right) \, dx + C \exp\left( -x \ln(x) + x \right) \]
**Existence & uniqueness**

\[ y' = q(x) - p(x) y \]

_q, p continuous on [a, b]_  
⇒ solution exists & is unique on (a, b)

4. Bernoulli's equation

\[ y' + p(x) y = q(x) y^n \quad n \neq 1 \]

Set \( u = y^{1-n} \)

\[
\frac{du}{dx} = (1-n)y^{-n} \quad y' = [q(x)y^n - p(x)y]y^{-n}(1-n) \\
= [q(x) - p(x)y^{1-n}](1-n) = [q(x) - p(x)u](1-n)
\]

i.e. \( u' + (1-n)p(x)u = q(x)(1-n) \), which is linear

**Example:** \( y' = ay - y^3 \)