Modeling with differential equations

Remarks:

1. Not all modeling is done with differential equations.

2. There needs to be a quantity that changes in time.

3. The dynamics should be deterministic (i.e. the rate of change can be expressed as a function of the state of the system, its parameters, etc)

4. The quantity we are interested in should be modeled by a continuous function of time.

5. When writing down a model, one starts with some "intuitive" description of what is going on in a given system.

6. All models have hypotheses, which are not going to be satisfied in some parameter regimes. Of course, the model is only valid in those regimes where the hypotheses are.

Example of mixture problem:

- independent variable: t, in minutes
- S = amount of salt, in pounds
- V = volume of mixture, in gallons
- C = concentration of salt in the mixture
  $C = \frac{S}{V}$ in pounds / gallon.

$$\frac{dS}{dt} = (\text{salt in}) - (\text{salt out})$$

$$= (\text{inflow})(\text{concentration of salt in mixture entering the tank})$$

$$- (\text{outflow})(\text{concentration of salt in mixture leaving the tank})$$

$$= 4 \cdot 3 - 5 \cdot C$$

$$= 12 - 5\frac{S}{V}$$

$$\frac{dV}{dt} = (\text{inflow}) - (\text{outflow})$$

$$= 4 - 5 = -1 \Rightarrow V(t) = -t + Cst$$

Use initial condition : at $t=0$, $V = 200$ gallons

So $V(t) = 200 - t$ gallons.

Thus $\dfrac{dS}{dt} = 12 - 5\dfrac{S}{200-t}$.

Solve this <u>linear</u> equation :

$$\frac{dS}{dt} + \frac{5S}{200-t} = 12$$

Multiply by $\exp\left(\int \dfrac{5\,dt}{200-t}\right)$

i.e. $\exp\left(-5 \ln|200-t|\right) = \exp\left(\ln \dfrac{1}{(200-t)^5}\right)$

$200-t = V(t)$
must be > 0

Therefore, we multiply by $\dfrac{1}{(200-t)^5}$,

Then, $\dfrac{1}{(200-t)^5}\dfrac{dS}{dt} + \dfrac{5S}{(200-t)^6} = \dfrac{12}{(200-t)^5}$

i.e. $\dfrac{d}{dt}\left(\dfrac{S}{(200-t)^5}\right) = \dfrac{12}{(200-t)^5}$

Integrate :

$$\frac{S}{(200-t)^5} = \int \frac{12}{(200-t)^5}\,dt$$

$$= \frac{3}{(200-t)^4} + \underbrace{Cst}_{K}$$

i.e. $S(t) = 3\,(200-t) + K\,(200-t)^5$

valid up to $t = 200$ mn.

Moreover, $C(t) = \dfrac{S(t)}{V(t)} = 3 + K\,(200-t)^4$

You can find K by saying that at $t=0$ $C(0) = 0$ or $S(0) = 0$.