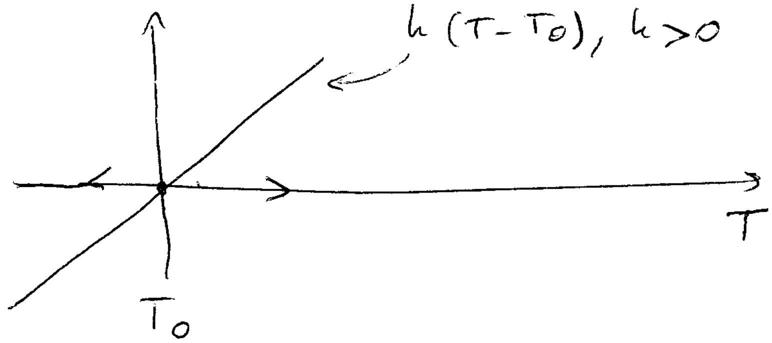
Modeling with differential equations (continued)

3. Cooling & heating

$$\frac{dT}{dt} = -k\,(T - T_0) \qquad k > 0$$

T_0 = ambient temperature



If k was negative, $T = T_0$ would be unstable, as shown above.
So choose $k > 0$.

Solve:
$$\frac{dT}{T - T_0} = -k\,dt$$

$$\ln|T - T_0| = -kt + C$$

$$\Rightarrow\; T - T_0 = \pm e^{C} e^{-kt} = \mathcal{K} e^{-kt}$$

i.e.
$$T = T_0 + \mathcal{K} e^{-kt} \qquad \mathcal{K} \in \mathbb{R}$$

4. Compounding interest (continuously)

M = amount of money in bank account

r = interest rate in %,
interest compounded continuously

$$\frac{dM}{dt} = \frac{r}{100} M$$

$$\exp(x) = \exp(0) + x\exp'(0) + \frac{x^2}{2}\exp''(0) + \dots$$
$$= 1 + x + \frac{x^2}{2} + \dots$$

$$\exp(x) - 1 = x + \frac{x^2}{2} + \dots$$

5. Population dynamics

N = density of peope in a given region.

$$\frac{dN}{dt} = \underset{\substack{\uparrow \\ \text{birth rate}}}{b}\, N - \underset{\substack{\uparrow \\ \text{death rate}}}{d}\, N - \text{immigration} + \text{emigration}$$

In the absence of immigration or emigration, we have

$$\frac{dN}{dt} = rN \qquad r = b - d = \text{growth rate (could be } < 0)$$

$$\frac{dN}{dt} = rN$$

$$\Rightarrow N = N_0 e^{rt}$$

When is $N = 2N_0$, assuming that $r>0$?

$$2N_0 = N_0 e^{rt} \Rightarrow \ln(2) = rt$$

$$\Rightarrow t = \frac{\ln(2)}{r}$$

$$t = \frac{\ln(2)}{r} = \text{doubling time}$$

If r is < 0, $N = N_0 e^{rt}$

Ask when $N = \frac{N_0}{2} = N_0 e^{rt}$

$$-\ln(2) = rt \Rightarrow t = \frac{\ln(2)}{-r} = \frac{\ln(2)}{|r|}$$

$\frac{1}{2}$ - life

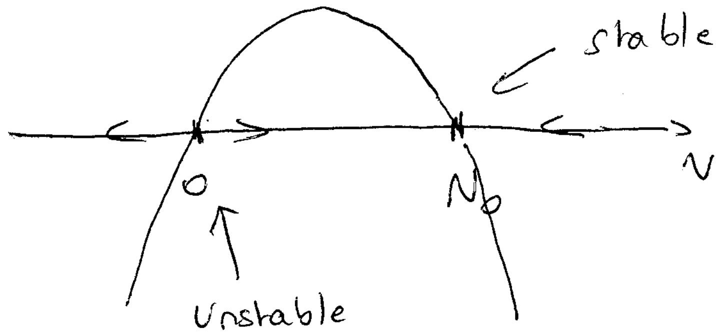
If resources are limited, growth will saturate (assume now $r>0$).

$$\frac{dN}{dt} = N\, r(N)$$

with $r(N)$ positive for N small (for growth) and negative for N large (for saturation)

$$\frac{dN}{dt} = rN\,(N_0 - N) = rN_0N - rN^2$$

$$N_0 > 0 \qquad r > 0$$



This is the logistic equation. To solve it, separate variables & integrate with partial fractions.

6. Chemical reactions

$$A + B \xrightarrow{k} C$$

$$\frac{d[A]}{dt} = -k\,[A]\,[B]$$

7. Tumor growth

$$\frac{dk}{dt} = -\alpha k \qquad k = k_0 e^{-\alpha t} = \beta e^{-\alpha t}$$

$$\frac{dN}{dt} = kN = \beta e^{-\alpha t} N$$

$$\frac{1}{N}\frac{dN}{dt} = \beta e^{-\alpha t} = \frac{d}{dt}\left(\ln(N)\right)$$

So $$\ln(N) = -\frac{\beta}{\alpha} e^{-\alpha t} + C$$
$$= -\frac{1}{\alpha}\frac{1}{N}\frac{dN}{dt} + C$$

At $t=0$ $N=1$ So $\ln(1) = -\frac{\beta}{\alpha} e^{-\alpha \cdot 0} + C \Rightarrow C = \frac{\beta}{\alpha}$

and $$\ln(N) = -\frac{1}{\alpha}\frac{1}{N}\frac{dN}{dt} + \frac{\beta}{\alpha} \Rightarrow \frac{1}{N}\frac{dN}{dt} = \beta - \alpha \ln(N)$$

i.e. $$\frac{dN}{dt} = N\left(\beta - \alpha \ln(N)\right)$$

which is the Gompertz equation.