Calculus and Differential Equations I

MATH 250 A

Introduction to differential equations

A simple differential equation

- Is there a function which is equal to its derivative?
  1. Yes
  2. No
- Is such a function unique?
  1. Yes
  2. No
- The equation $\frac{dy}{dx} = y$ is an example of an ordinary differential equation.
- The independent variable is x and the dependent variable is y.
- The above equation is a first order, autonomous and linear differential equation.

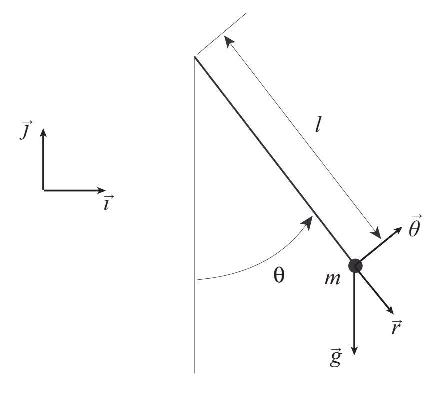
Solutions of a differential equation

- An explicit solution of an ordinary differential equation

$$\frac{dy}{dx} = f(x, y)$$

  is a function $y(x)$ such that when substituted into the differential equation, both sides are found to be identical.
- In general, a given differential equation will have a family of solutions, involving one or more parameters.
- Applying initial or boundary conditions often leads to the selection of one of these solutions.
- We will now turn to two important questions: what are differential equations used for, and how do we study them?
- To address the first questions, we now look at examples of differential equations or systems thereof.

The nonlinear pendulum



Sketch of a point-mass pendulum

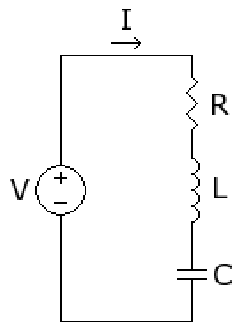
The equation of motion for the nonlinear pendulum is given by

$$m\,l\,\frac{d^2\theta}{dt^2} = -m\,g\,\sin(\theta) - c\,l\,\frac{d\theta}{dt},$$

where

- θ and t are variables.
- m, l, g and c are parameters.
- Most of these quantities are defined on the figure, except c, which measures friction.

The RLC circuit



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The series RLC circuit consists of a resistor of resistance R, an inductor of inductance L, a capacitor of capacitance C, and a power source of voltage $V(t)$.

The charge q across the capacitor satisfies the differential equation

$$L\,\frac{d^2q}{dt^2} + R\,\frac{dq}{dt} + \frac{1}{C}\,q = V(t),$$

and the current in the circuit is given by $I(t) = \dfrac{dq}{dt}$.

The classic SIR model

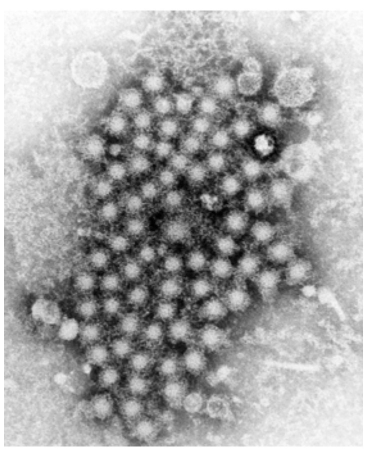
Penned goats in a village within a region investigated for a Rift Valley fever outbreak in Saudi Arabia. Picture # 8362, Public Health Image Library

The classic SIR model reads

$$\begin{aligned}\frac{dS}{dt} &= -\alpha\,S\,\frac{I}{N},\\ \frac{dI}{dt} &= \alpha\,S\,\frac{I}{N} - \beta\,I,\\ \frac{dR}{dt} &= \beta\,I,\end{aligned}$$

where S, I, and R represent the numbers of susceptible, infectious, and recovered (or removed) individuals, in a population of size N. The parameter α measures the average number of positive contacts per susceptible per unit of time, and β measures the rate at which individuals recover.

Viral infections



Transmission electron micrograph showing hepatitis virions of an unknown strain. Picture # 8153, Public Health Image Library

The the dynamics of a viral infection, such as hepatitis B or C, may be described by the following model (M.A. Nowak *et al.*, Proc. Natl. Acad. Sci. USA **93**, 4398-4402 (1996)).

$$\begin{aligned}\frac{dX}{dt} &= \lambda - \delta\,X - b\,V\,X\\ \frac{dY}{dt} &= b\,V\,X - a\,Y\\ \frac{dV}{dt} &= k\,Y - \kappa\,V\end{aligned}$$

The variable X represents the number of uninfected cells, Y is the number of infected cells, and V is the viral load (or number of free virions in the body).

How do we study differential equations?

- Sometimes, we can solve a differential equation. In this class (MATH 250 A & B), we will learn how to solve first and second order linear equations and systems of first order linear equations, as well as some first order nonlinear equations.
- If initial conditions are known, one can solve a differential equation (or a system of differential equations) numerically. We will learn a simple numerical method to solve a differential equation and also use more advanced algorithms in MATLAB.
- Before trying to solve a differential equation, or launching into a numerical exploration of its properties, one needs to know whether solutions exist and if so, whether they are unique. We will see theorems that guarantee existence and uniqueness of solutions to differential equations.

How do we study differential equations? (continued)

- In many situations, especially when one deals with nonlinear differential equations, one cannot find explicit solutions.
- In this case, one can nevertheless understand the dynamics of a differential equations by looking at special solutions and at their stability.
- The qualitative theory of dynamical systems (discussed in MATH 454) provides a way of understanding the behavior of a system of differential equations, as well as the bifurcations that occur when one or more parameters are changed. We will briefly address some of these issues.
- Partial differential equations (discussed in MATH 322, MATH 422, and MATH 456) are differential equations describing the dynamics of systems with two or more independent variables.

What we will do next

- We will start with the simplest type of differential equations, $\frac{dy}{dx} = g(x)$.
  - Chapter 1 of *Differential Equations* book. Reading assignment: Sections 1.1, 1.2, and 1.3.
  - Solving such differential equations involves integration, so we will introduce various methods of integration discussed in the *Calculus* book.
- We will then consider autonomous differential equations of the form $\frac{dy}{dx} = g(y)$.
  - Chapter 2 of the *Differential Equations* book.
  - Ideas of stability and bifurcations.
  - One more technique of integration: partial fractions.

What we will do next (continued)

- We will then turn to general first order differential equations, $\frac{dy}{dx} = g(x, y)$
  - Chapter 3 of the *Differential Equations* book.
  - Graphical analysis, symmetries, scalings, and numerical solutions.
  - Existence and uniqueness of solutions.
- Finally, we will look at various methods of solution for first order differential equations
  - Chapters 4 and 5 of the *Differential Equations* book.
  - Separation of variables and equations with homogeneous coefficients.
  - First order linear differential equations and Bernouilli's equation.