Newton’s law: for an object moving in one dimension

\[ F = m \gamma = m \frac{dv}{dt} , \]

where \( F \) is the sum of forces applied along the positive \( x \) direction, \( m \) is the mass of the object, \( x \) is the position of its center of mass, and \( v = \frac{dx}{dt} \) is its velocity.

- If the only force is gravity, then \( F = -mg \) if \( x \) points upward in the vertical direction. In this case, we have

\[ \frac{dv}{dt} = -g , \]

which is solved by direct integration.

Objects in motion (continued)

- In the presence of gravity and friction, we typically have
  - \( F = -mg - cv, \) \( c > 0, \) if the object is moving slowly.
  - \( F = -mg + cv^2, \) \( c > 0, \) if \( |v| \gg 1, \) \( v < 0. \)
  - It is possible to have other types of friction forces, especially in the case of solid friction.

- If we restrict ourselves to the above examples, we have
  - \( \frac{dv}{dt} = -g - \frac{c}{m} v, \) which is linear in \( v. \)
  - \( \frac{dv}{dt} = -g + \frac{c}{m} v^2, \) which is separable.

- For a spring-mass system, we have \( F = -k(x - x_0), \) \( k > 0. \)
  Then, \( m \frac{d^2x}{dt^2} = -k(x - x_0), \) which is a second order, linear equation.

Mixture problems

- These problems typically involve a fluid, of volume \( V(t), \) in which a substance is dissolved. The goal is to find the amount \( A(t) \) or the concentration \( C(t) = A(t)/V(t) \) of the substance in the fluid.

- The general way of addressing such a problem is to write a balance equation for the amount \( A(t) \) of the substance in the fluid,

\[ \frac{dA}{dt} = \text{input rate} - \text{output rate} \]

- Example (#5 page 207): Take a 200-gallon container filled with pure water. Add a salt concentration with 3 pounds of salt per gallon, at a rate of 4 gallons per minute. At the same time, drain the container at a rate of 5 gallons per minute. Find the amount of salt in the container as a function of time.
Cooling and heating

- Newton’s law of cooling and heating says that the rate of change of the temperature $T$ of an object is a linear function of the difference between $T$ and the ambient temperature $T_0$:
  $$\frac{dT}{dt} = -k(T - T_0), \quad k > 0.$$  

- This equation can be solved as a linear equation, or as a separable equation, to find
  $$T(t) = T_0 + \kappa \exp(-kt),$$
  where $\kappa$ is an arbitrary constant.
- As expected, $T \to T_0$ as $t \to +\infty$.

Compounding interest

- If money in a bank account is compounded continuously at a rate of $r$ percents per year, then in the absence of deposits or withdrawals, we have
  $$\frac{dM}{dt} = \frac{r}{100} M,$$
  where $M$ is the account balance and $t$ is time measured in years.
- The above equation describes the exponential growth of $M$.
- After one year, the amount of money in the account is given by
  $$M(1) = \exp\left(\frac{r}{100}\right) M(0).$$
- The annual interest rate is therefore larger than $r/100$, since
  $$APY = \exp\left(\frac{r}{100}\right) - 1.$$  

Population dynamics

- If $N$ is the population density of a region, then one can write
  $$\frac{dN}{dt} = bN - dN + \text{immigration} - \text{emigration},$$
  assuming that resources are not limited.
- In the above equation, $b$ is the birth rate, and $d$ is the death rate of the population. The growth rate $r$ of the population is given by
  $$r = b - d.$$  
- If immigration and emigration are given functions of $t$, then the above equation is linear in $N$.

Population dynamics (continued)

- If a population is growing exponentially at rate $r > 0$, we can define its doubling time
  $$T_d = \frac{\ln(2)}{r}.$$  
- Note the analogy with the half-life of a substance decaying exponentially at rate $r < 0$,
  $$T_{1/2} = -\frac{\ln(2)}{r} = \frac{\ln(2)}{|r|}.$$  
- If resources are limited, one can expect that $r$ will depend on $N$. With $r = \alpha - \beta N$, $\alpha > 0$, $\beta > 0$, and in the absence of immigration or emigration, we have logistic growth
  $$\frac{dN}{dt} = \alpha N - \beta N^2.$$
Chemical reactions

- For a chemical reaction of the form $A + B \xrightleftharpoons[k_2]{k_1} C$, the law of mass action says that

$$\frac{d[C]}{dt} = k_1[A][B] - k_2[C],$$

where $[X]$ is the concentration of chemical $X$ and $k_1$ and $k_2$ are the forward and backward rate constants respectively.

- For an autocatalytic reaction of chemical $X$, one may have

$$\frac{d[X]}{dt} = k_1a[X] - k_2[X]^2,$$

where $a$, $k_1$, and $k_2$ are constants. This is again the logistic equation.