Differential equations of the form $y' = g(x)$

**Formal solutions (continued)**

- We can therefore write the general solution to $y' = g(x)$ as
  
  $$y(x) = \int_a^x g(t) \, dt + C.$$  

- If we can find an antiderivative of $g(x)$, then we have an explicit solution.

- Given a continuous function $g$, can we always find an explicit expression for an antiderivative of $g$?
  - Yes
  - No

- There exists functions which we do not know how to integrate. An example is the error function
  
  $$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) \, dt.$$  

**Formal solutions**

- The differential equation $y' = g(x)$, where $g$ is continuous, may formally be solved by integration, so that
  
  $$y(x) = \int g(x) \, dx + C.$$  

- **Fundamental theorem of Calculus**: if $f$ is a continuous function on $[a, b]$ and if $f(x) = F'(x)$, then
  
  $$\int_a^b f(x) \, dx = F(b) - F(a).$$  

- **Second fundamental theorem of Calculus**: if $f$ is continuous on $[a, b]$ and if $x \in [a, b]$, then
  
  $$\frac{d}{dx} \int_a^x f(t) \, dt = f(x).$$  

- Even if we do not know how to integrate $g$, we may still be able to say something about the behavior of $y(x)$ by looking at properties of the antiderivative of $g$.

- As $x \to \infty$, the integral in the expression for $y$ becomes an improper integral of the form $\int_a^\infty g(t) \, dt$.

  We will study improper integrals in the second semester of this course.

- An initial or boundary condition of the form $y(x_0) = y_0$ allows us to pick a particular solution from the one-parameter family of solutions:
  
  $$y(x) = \int_{x_0}^x g(t) \, dt + y_0.$$  

- A solution curve is the graph of a particular solution $y(x)$. 

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Differential equations of the form $y' = g(x)$  
Calculus and Differential Equations I
Existence and uniqueness of solutions

- **True/False:** All solution curves of \( y' = g(x) \) may be obtained by vertical translation of one of them.
  - True
  - False

- Since we assume that \( g \) is continuous, we know that solutions exist. Are they unique?
  - Yes
  - No

- If there is a unique solution for any initial condition, is it possible for solution curves to cross (or meet) at a point?
  - Yes
  - No

- If at some point \( g \) becomes singular, e.g. if it is undefined or it stops being continuous, the argument for existence, which assumes that \( g \) is continuous, will fail.

- Nevertheless, in some cases it will be possible to patch solutions of a differential equation found in adjacent intervals.

Qualitative properties of solutions

- Since we know \( y' \), and assuming that \( g \) is smooth, we know all of the derivatives of \( y \).

- As a consequence, we know when the solution \( y \) is increasing or decreasing, and we know the concavity of the graph of \( y \).

- We can use symmetries of the equation to relate a solution to another solution. For instance, if \( y(x) \) is a solution and \( g(x) \) is odd, then \( u(x) = y(-x) \) is also a solution.

- If \( y(x) \) is a solution and \( g(x) \) is even, then which of the solutions below is also a solution?
  - \( u(x) = y(-x) \)
  - \( u(x) = -y(-x) \)
  - None of the above

- Note that symmetries tell you about properties of the family of solutions, not of each particular solution.

Example of application

Consider the differential equation \( y' = \frac{x^2 + 1}{x^2 - 1} \).

- **True/False:** Solution curves increase for \( x \) in \((-1,1)\).
  - True
  - False

- **True/False:** Solution curves are concave up for \( x \geq 0 \).
  - True
  - False

- **True/False:** Isoclines are all parallel to the \( y \)-axis.
  - True
  - False

- **True/False:** The family of solution curves is symmetric with respect to the \( y \)-axis.
  - True
  - False

Slope fields

A slope field for the differential equation \( y' = g(x) \) is a collection of line segments in the \((x,y)\) plane such that the slope of the segment centered at point \((x,y)\) is equal to \( g(x) \).

- Given an initial condition \( y(x_0) = y_0 \), one can sketch the associated solution curve (assuming it exists and is unique) by following the slope field, starting from \((x_0,y_0)\).

- You should enter the slope field program into your calculator. In class, we will use PPLANE to plot slope fields.
Some of your questions

Uniqueness
- Why is it important?
- Can we see an example of a differential equation for which solutions are not unique?
- Are singularities in $g(x)$ related to uniqueness issues?
- How is uniqueness related to non-intersecting solution curves?

Solutions
- Do differential equations always have an infinite number of solutions?
- If so, do they always differ by a constant?
- How do we know when a differential equation cannot be solved?
- What happens to solutions of $y' = g(x)$ as $x$ goes to infinity?
- Can an explicit solution have a vertical tangent?
- If $g(x)$ has a vertical tangent at $x_0$, does it mean that if one solves $dx/dy = 1/g(x)$, one would have an horizontal tangent at that point?
- If we solve $dy/dx = g(x)$ and $dx/dy = 1/g(x)$, do we get the same solution curves?

Slope fields
- How far in $x$ and $y$ should we go?
- How far apart should the points in the $x$-$y$ plane be?
- What do isoclines mean?
- Why are slope fields useful / necessary?
- How do we identify a singularity in a slope field?
- If there exists isoclines of infinite slope, can one construct a solution that spans both sides of the isocline?
- Why can’t we just use slope fields to solve ode’s?