Math 250A (Fall 2008) - Exam I **Name:**

Instructions:

- Show all work clearly in order to get full credit. Points can be taken off if it is not clear to see how you arrived at your answer (even if the final answer is correct).
- When you do use your calculator, sketch all relevant graphs and explain all relevant mathematics.
- Circle your final answers.
- Please keep your written answers brief; be clear and to the point.
- This test has 6 problems (plus an extra credit problem) and is worth 100 points, plus extra credit at the end. It is your responsibility to make sure that you have done all the problems!

1. (*15 points*) Find the following derivatives. Make sure to simplify your answers.

(a) $\frac{d}{dx} \ln\left(\frac{x}{a+x}\right)$ where a is a constant

$$\frac{d}{dx} \ln\left(\frac{x}{a+x}\right) = \frac{\frac{a+x-x}{(a+x)^2}}{\frac{x}{a+x}} = \boxed{\frac{a}{x(a+x)} = \frac{d}{dx}\ln\left(\frac{x}{a+x}\right)}$$

(b) $\frac{d}{d\alpha}\left[\frac{1}{3}\alpha^3 + \alpha\, 7^{2\alpha}\right] = \frac{d}{d\alpha}\left[\frac{1}{3}\alpha^3 + \alpha\, e^{2\alpha \ln(7)}\right]$

$$= \alpha^2 + e^{2\alpha \ln(7)} + \alpha\, 2\ln(7)\, e^{2\alpha\ln(7)}$$

$$= \boxed{\alpha^2 + 7^{2\alpha}\left[1 + 2\alpha \ln(7)\right] = \frac{d}{d\alpha}\left[\frac{1}{3}\alpha^3 + \alpha\, 7^{2\alpha}\right]}$$

(c) $\frac{d}{d\phi}\frac{\cos(\phi^3)}{\phi^2} = -2\frac{\cos(\phi^3)}{\phi^3} - \frac{3\phi^2 \sin(\phi^3)}{\phi^2} = -\frac{2}{\phi^3}\cos(\phi^3) - 3\sin(\phi^3)$

$$\boxed{\frac{d}{d\phi}\left(\frac{\cos(\phi^3)}{\phi^2}\right) = -\frac{2}{\phi^3}\cos(\phi^3) - 3\sin(\phi^3)}.$$

2. (*10 points*) Suppose that a large oil tanker is ocean-bound and strikes another ship as it leaves harbor, causing a rupture in the hull. Oil starts leaking at a rate $r = f(t)$ gallons per minute, where t is in minutes. After 2.5 hours, the leak is completely patched up. The total amount of oil leaked by the tanker is given by

$$Q = \int_0^x f(t)\, dt.$$

(a) What are the units of Q? Why?

$$Q = \int_0^x f(t)\, dt$$

$f(t)$ is in gallons per minute
dt is in minutes so

$\boxed{Q \text{ is in gallons.}}$

(b) What should the upper bound x be? What are its units?

$x = 2.5 \times 60 = \boxed{150 \text{ minutes} = x}$.

x is the time at which the leak is patched up, expressed in minutes.

(c) Suppose that $r(t) = 750e^{-t/\tau}$ where $\tau = 240\,\text{min}$. Calculate the total number of gallons leaked.

$$Q = \int_0^x 750\, e^{-t/\tau}\, dt = 750\left[-\tau e^{-t/\tau}\right]_0^x$$

$$= 750\,\tau\left[1 - e^{-x/\tau}\right]$$

$$= \boxed{750 \cdot 240\left[1 - e^{-150/240}\right] = Q}$$

$\simeq 83652.94$ gallons.

3. (*15 points*) A frozen biological sample is taken out of the freezer and left on the lab counter at room temperature. The temperature (T, in degrees Fahrenheit) of the sample is given by $T = f(t)$, where t is the time in hours since the sample was removed from the freezer.

(a) What is the sign of $f'(t)$? Why?

$\boxed{f'(t) > 0}$ since the sample is warming up.

(b) What are the units of $f'(2)$? Why?

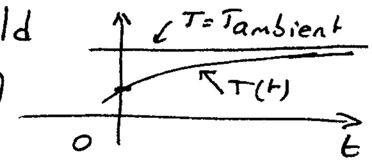
$f'(2) = \frac{dT}{dt}(2)$ is in $\boxed{\text{degrees Farenheit per hour}}$

(c) What is the practical meaning of the statement $f'(2) = 0.63$?

When $t = 2$ hours, the temperature changes at a rate of 0.63 °F per hour.

(d) What do you think the sign of $f''(2)$ is? Explain.

Since we expect T to saturate near ambient temperature the graph of T as a function of time should be concave down, so $\boxed{f''(2) \text{ should be negative.}}$



4. (*15 points*) Consider the differential equation

$$\frac{dy}{dx} = \frac{xy}{x^2 - 1}$$

(a) Is this equation linear? Why or why not?

Since the right-hand side does not contain any power or nonlinear function of y, this equation is $\boxed{\text{linear}}$.

(b) Is this equation autonomous? Why or why not?

The right-hand side contains x explicitly. So the equation is $\boxed{\text{non-autonomous}}$.

(c) Give an example of a 2nd order, nonlinear, non-autonomous differential equation. Explain why the equation you wrote satisfies these conditions.

$\boxed{\frac{d^2y}{dx^2} = y^2 x}$ is second-order (because the 2nd derivative is the derivative of highest order), nonlinear (because of y^2) and non-autonomous because of the x in the right-hand side.

5. (*20 points*) Consider the differential equation

$$\frac{dy}{dx} = \frac{1}{x^2} + 0.5$$

(a) Write down a general solution of this equation in the form of an integral.

Integrate both sides of the equation to get

$$y(x) = \int_a^x \left(\frac{1}{t^2} + 0.5\right) dt + C \quad \text{where } a \neq 0$$

Equivalently, one can write $\boxed{y(x) = \int \left(\frac{1}{x^2} + 0.5\right) dx + C.}$

(b) Evaluate the integral to obtain an explicit family of solutions.

$\boxed{y(x) = \frac{-1}{x} + 0.5x + C}$ by integration.

(c) Find the particular solution that passes through the point $P = (1, 3)$.

When $x = 1$, $y(3)$ so $3 = y(1) = \frac{-1}{1} + 0.5 \cdot 1 + C = -0.5$

$\Rightarrow C = 3.5$

Thus, the particular solution that goes through point P is $\boxed{y(x) = -\frac{1}{x} + 0.5x + 3.5}$.

(d) What happens to the solution found in part (c) as $x \to 0$?

Note that the solution is only defined for $x > 0$.

As $\boxed{x \to 0^+, \quad y(x) \to -\infty}$ (because of the term in $\frac{-1}{x}$).

6. (*25 points*) Consider the following function

$$y(x) = 3x^4 - 24x^2 + 8x^3 - 96x + 10$$

(a) $x = 2$ is a critical point of this function. Find all other critical points, if any.

A critical point is such that $y'(x) = 0$ or $y'(x)$ is undefined.
Here since y is a polynomial, the critical points are given by $y'(x)=0$.

$y'(x) = 12x^3 - 48x + 24x^2 - 96 = 12\left[x^3 - 4x + 2x^2 - 8\right]$

$y'(x) = 0 \Leftrightarrow x^3 - 4x + 2x^2 - 8 = 0 \Leftrightarrow (x-2)(x^2 + 4x + 4) = 0$

$\Leftrightarrow (x-2)(x+2)^2 = 0 \Leftrightarrow x = 2$ or $x = -2$

So there is <u>one other critical point</u> at $\boxed{x=-2}$.

(b) Find all the inflection points of the function.

An inflection point is such that y'' vanishes & changes sign at the inflection point.

$y''(x) = \frac{d}{dx}\left\{12(x^3 - 4x + 2x^2 - 8)\right] = 12(3x^2 - 4 + 4x)$

$= 12(x+2)(3x-2)$

$y''(x) = 0 \Leftrightarrow x = -2$ or $x = \frac{2}{3}$.

Since $y''(x)$ is proportional to the product $(x+2)$ times $(3x-2)$, y'' changes sign at $x=-2$ and $x = \frac{2}{3}$. So there are 2 <u>inflection points</u>, $\boxed{x = -2 \text{ and } x = 2/3}$

(c) Find the minima and maxima of the function, if any. For each extremum found, indicate whether it is a local or global extremum. Explain.

$y'(x) = 12(x-2)(x+2)^2$ so y' does not change sign at $x=-2$. It however goes from negative to positive as x crosses the value $x = 2$. Therefore, y has a <u>minimum at $x=2$</u>.

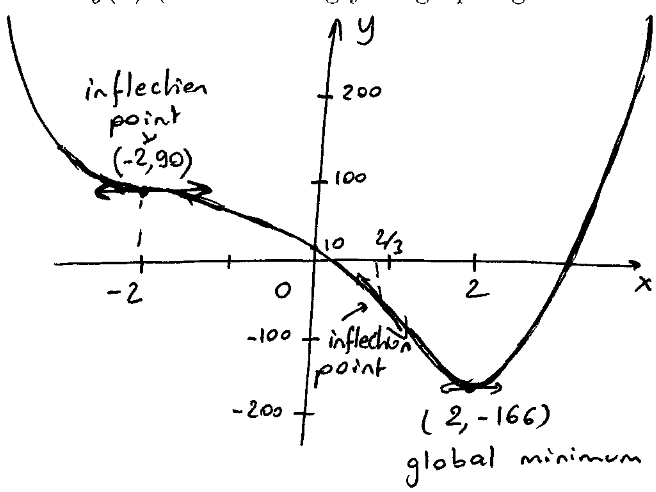
Since $\lim_{x \to \pm\infty} y(x) = +\infty$, and since $x=2$ is the sole extremum, it is also a <u>global minimum</u>.

(d) What is $\lim_{x \to \pm\infty} y(x)$?

$$\lim_{x \to \pm\infty} y(x) = \lim_{x \to \pm\infty} \left[3x^4 - 24x^2 + 8x^3 - 96x + 10 \right]$$

$$= \lim_{x \to \pm\infty} (3x^4) = \boxed{+\infty = \lim_{x \to \pm\infty} y(x)}$$

(e) Sketch $y(x)$ (without using your graphing calculator!). Make sure to label your axes.



$$y(2) = 3 \cdot 16 - 24 \cdot 4 + 8 \cdot 8 - 96 \cdot 2 + 10$$
$$= 48 - 96 + 64 - 192 + 10$$
$$= -48 + 64 - 182 = 16 - 182$$
$$= -166$$

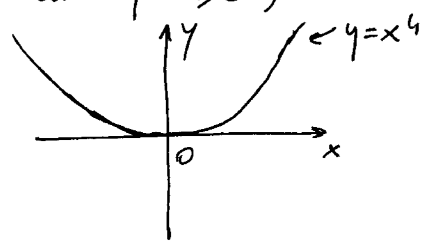
$$y(-2) = 3 \cdot 16 - 24 \cdot 4 - 8 \cdot 8 + 96 \cdot 2 + 10$$
$$= 48 - 96 - 64 + 192 + 10$$
$$= -48 - 64 + 202 = -112 + 202$$
$$= 90$$

$$y(0) = 10$$

(f) True of False: A function has an inflection point at $x = x_o$ if $f''(x_o) = 0$. Justify your answer.

False. f'' needs to change sign at x_o for x_o to be an inflection point.

A counter-example is as follows: Let $f(x) = x^4$ and $x_o = 0$. We have $f'(x) = 4x^3$ and $f''(x) = 12x^2$. So $f''(0) = 0$ but the graph of f (which is always ≥ 0) does not cross its tangent at $x = 0$



Extra Credit (6 Points):

(a) For an arbitray function $f(t)$, find $\dfrac{d}{dx}\displaystyle\int_0^{\cos(x)} f(t)\,dt$

$$\frac{d}{dx}\left[\int_0^{\cos(x)} f(t)\,dt\right] = f(\cos(x))\,\frac{d}{dx}(\cos(x))$$

where we have used the chain rule and the fundamental theorem of calculus $\left(\frac{d}{dx}\int_0^{x} f(t)\,dt = f(x)\right)$.

Since $\frac{d}{dx}(\cos(x)) = -\sin(x)$, we have

$$\boxed{\frac{d}{dx}\int_0^{\cos(x)} f(t)\,dt = -\sin(x)\,f(\cos(x))}.$$

(b) Suppose $f(t) = \sin(t)$. Find $\dfrac{d}{dx}\displaystyle\int_0^{\cos(x)} f(t)\,dt$

Since $f(t) = \sin(t)$, the above formula gives

$$\boxed{\frac{d}{dx}\int_0^{\cos(x)} \sin(t)\,dt = -\sin(x)\sin(\cos(x)).}$$