**Math 250A (Fall 2008) - Exam II** **Name:**

**Instructions:**

- Show all work clearly in order to get full credit. Points can be taken off if it is not clear to see how you arrived at your answer (even if the final answer is correct).
- When you do use your calculator, sketch all relevant graphs and explain all relevant mathematics.
- A table of integrals is attached. If you use a formula from the table, indicate which one it is that you used.
- Circle your final answers.
- Please keep your written answers brief; be clear and to the point.
- This test has 5 problems (plus an extra credit problem) and is worth 100 points, plus extra credit at the end. It is your responsibility to make sure that you have done all the problems!

**1.** (*20 points*) Find the following integral: $\int_{1/3}^{2} y \ln(3y)\, dy$

Do not use the table of integrals and show all your work.

Set $u = 3y$, then $du = 3\,dy$ and

$$\int_{1/3}^{2} y \ln(3y)\, dy = \int_{1}^{6} \frac{u}{3} \ln(u) \frac{du}{3} = \frac{1}{9}\int_{1}^{6} u \ln(u)\, du$$

Now integrate by parts. $U' = u$ $\quad U = \frac{1}{2}u^2$

$V = \ln(u)$ $\quad V' = \frac{1}{u}$

So $\frac{1}{9}\int_{1}^{6} u \ln(u)\, du = \frac{1}{9}\left[\left[\frac{1}{2}u^2 \ln(u)\right]_1^6 - \int_1^6 \frac{1}{2}u^2 \frac{1}{u}\, du\right]$

$$= \frac{1}{9}\left[\left[\frac{1}{2}u^2 \ln(u)\right]_1^6 - \int_1^6 \frac{u}{2}\, du\right] = \frac{1}{9}\left[\frac{1}{2}u^2 \ln(u) - \frac{u^2}{4}\right]_1^6$$

$$= \frac{1}{9}\left[\frac{1}{2}36 \ln(6) - \frac{36}{4} + \frac{1}{4}\right] = \boxed{2\ln(6) - \frac{35}{36} \simeq 2.611}$$

**2.** (*10 points*) Solve the differential equation for $x(t)$: $\dfrac{dx}{dt} = \dfrac{1}{\sqrt{9t^2+25}}$

[Hint: You may need to use the table of integrals. If you do, make sure you explain which formula you used and how you used it.]

$$\frac{dx}{dt} = \frac{1}{\sqrt{9t^2+25}} \Rightarrow x(t) = \int \frac{dt}{\sqrt{9t^2+25}} + C.$$

$$\int \frac{dt}{\sqrt{9t^2+25}} = \int \frac{dt}{\sqrt{(3t)^2+25}}$$ Let $u = 3t$. Then $du = 3dt$

and $$\int \frac{dt}{\sqrt{9t^2+25}} = \int \frac{du}{3\sqrt{u^2+25}} = \frac{1}{3}\int \frac{du}{\sqrt{u^2+5^2}}$$

Using formula VI.29 of the table with $a = 5$, we have

$$\int \frac{du}{\sqrt{u^2+5^2}} = \ln\left|u + \sqrt{u^2+5^2}\right| + C = \ln\left|3t + \sqrt{9t^2+25}\right| + C$$

So $$\boxed{x(t) = \frac{1}{3}\ln\left|3t + \sqrt{9t^2+25}\right| + C}$$

**3.** (*15 points*) Solve the differential equation for $h(\theta)$: $\dfrac{dh}{d\theta} = \sin 3\theta(\cos 3\theta)^5$

Do not use the table of integrals and show all your work.

$$\frac{dh}{d\theta} = \sin(3\theta)\cos^5(3\theta) \Rightarrow h(\theta) = \int \sin(3\theta)\cos^5(3\theta)\,d\theta + C$$

Let $u = \cos(3\theta)$ $\quad du = -3\sin(3\theta)\,d\theta$

So $$\int \sin(3\theta)\cos^5(3\theta)\,d\theta = \int u^5 \frac{-du}{3} = -\frac{1}{3}\int u^5\,du$$

$$= -\frac{1}{18}u^6 + \tilde{C} = -\frac{1}{18}[\cos(3\theta)]^6 + \tilde{C}$$

Therefore $\boxed{h(\theta) = -\frac{1}{18}[\cos(3\theta)]^6 + K}$, where $K$ is an arbitrary constant.

**4.** (*45 points*)
Consider the differential equation

$$\frac{dy}{dt} = \frac{3-y}{2}$$

(a) (5 points) For which values of $y$ are solutions increasing? For which values of $y$ are they decreasing? Explain your reasoning.

Solutions increase when $\frac{dy}{dt} > 0$, i.e. for $3-y>0$,

i.e. for $\boxed{y<3}$.

They derease for $\frac{dy}{dt} < 0$, i.e. $\boxed{y>3}$.

(b) (5 points) For which values of $y$ are solution curves concave up? For which values of $y$ are they concave down? Explain your reasoning.

Solution curves are concave up if $\frac{d^2y}{dt^2} > 0$, and concave down for $\frac{d^2y}{dt^2} < 0$.

Now, $\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{3-y}{2}\right) = -\frac{1}{2}\frac{dy}{dt} = -\frac{1}{2}\left(\frac{3-y}{2}\right) = \frac{y-3}{4}$

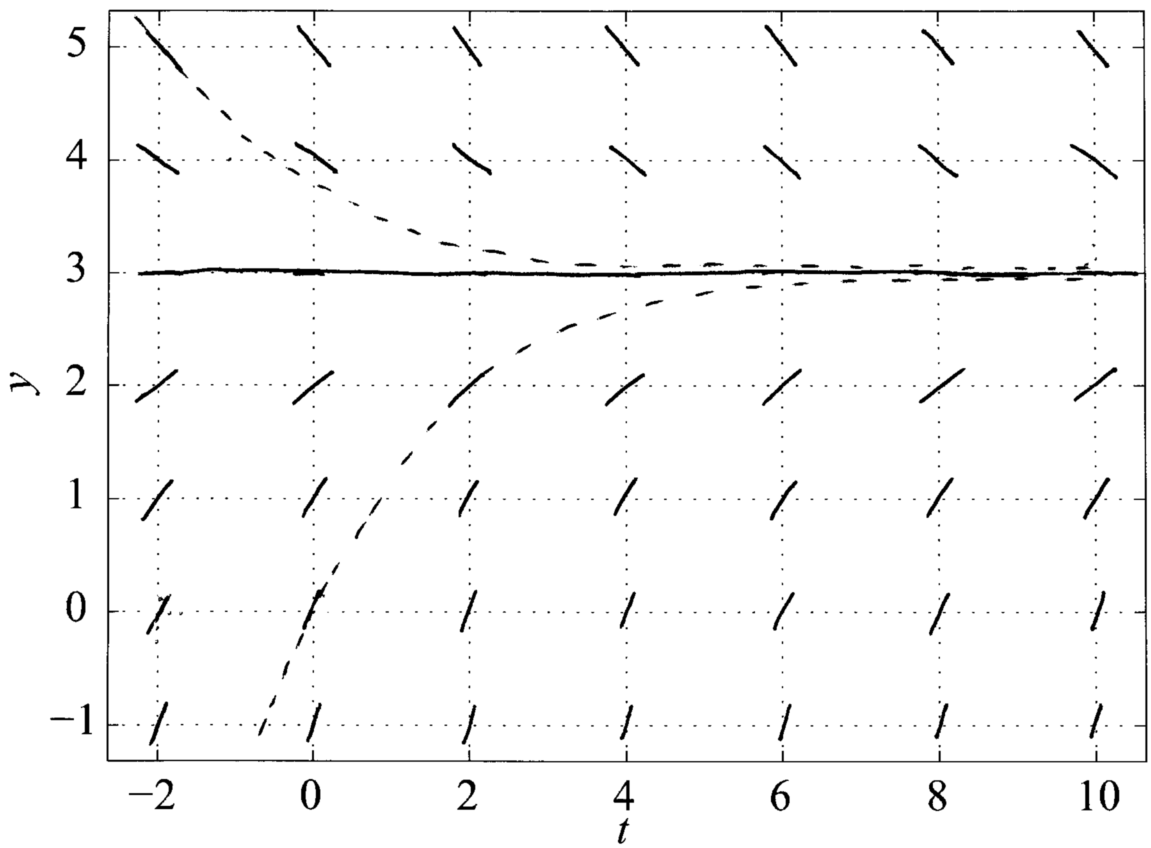
So solution curves are $\boxed{\begin{array}{l}\text{concave up for } y>3 \\ \text{concave down for } y<3\end{array}}$.

(c) (5 points) (i) True/False: The family of solution curves is symmetric with respect to the origin.

(ii) Explain your reasoning: If the family of solutions were symmetric with respect to the origin, then the equation would be unchanged if one changed $t$ into $-t$ and $y$ into $-y$. This is not the case here.

[This problem is continued on the next page]

(d) (15 points) (i) Plot the slope field in the figure below and draw 3 (different) solution curves of your choice.




(ii) List below what you think are important features of the slope field (e.g. nature of isoclines, equilibrium solutions, etc)

$\frac{dy}{dt} = \frac{3-y}{2}$. Isoclines are such that $\frac{3-y}{2} = m$, i.e. $y = 3-2m$. They are horizontal lines.

$y=3$ is an equilibrium solution so the slope field is made of segments parallel to the t-axis on the line $y=3$.

Table of slopes :

| $y$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ |
|---|---|---|---|---|---|---|---|
| $y'$ | $2$ | $3/2$ | $1$ | $1/2$ | $0$ | $-1/2$ | $-1$ |

[This problem is continued on the next page]

(e) (10 points) Find an analytic solution to the differential equation for which $y(0) = 1$. Show all your work.

$$\frac{dy}{dt} = \frac{3-y}{2} \Rightarrow \frac{dy}{3-y} = \frac{dt}{2}$$

$$\Rightarrow -\ln|y-3| = \frac{t}{2} + C$$

$$\Rightarrow |y-3| = e^{-\frac{t}{2}-C}$$

$$\Rightarrow y-3 = \pm e^{-C} e^{-t/2} = K e^{-t/2}$$

$$\Rightarrow y = 3 + K e^{-t/2}$$

Set $y(0) = 1$ to find $K$:

$$1 = 3 + K e^{-0/2} = 3 + K \Rightarrow K = -2$$

So $\boxed{y(t) = 3 - 2e^{-t/2}}$.

Note that as $t \to +\infty$, $y(t) \to 3$.

(f) (5 points) Are there any equilibrium solutions? If so, what are they? How do you find them?

Equilibrium solutions are such that $\frac{dy}{dt} = 0$, i.e.

$\frac{3-y}{2} = 0 \Leftrightarrow \boxed{y = 3}$.

So there is 1 equilibrium solution.

**5.** (*10 points*)
(a) (5 points) Give an example of a differential equation whose family of solution curves is symmetric with respect to the origin. Explain why you think the differential equation in question has this particular symmetry.

Consider the differential equation $\frac{dy}{dx} = x^2$.

If $y(x)$ is a solution, then let $u(x) = -y(-x)$. We have
$\frac{du}{dx} = -\frac{d}{dx}(y(-x)) = +y'(-x) = (-x)^2 = x^2$,
so $u(x)$ is also a solution. Therefore, changing $x$ into $-x$ and $y$ into $-y$ leaves the family of solutions unchanged.

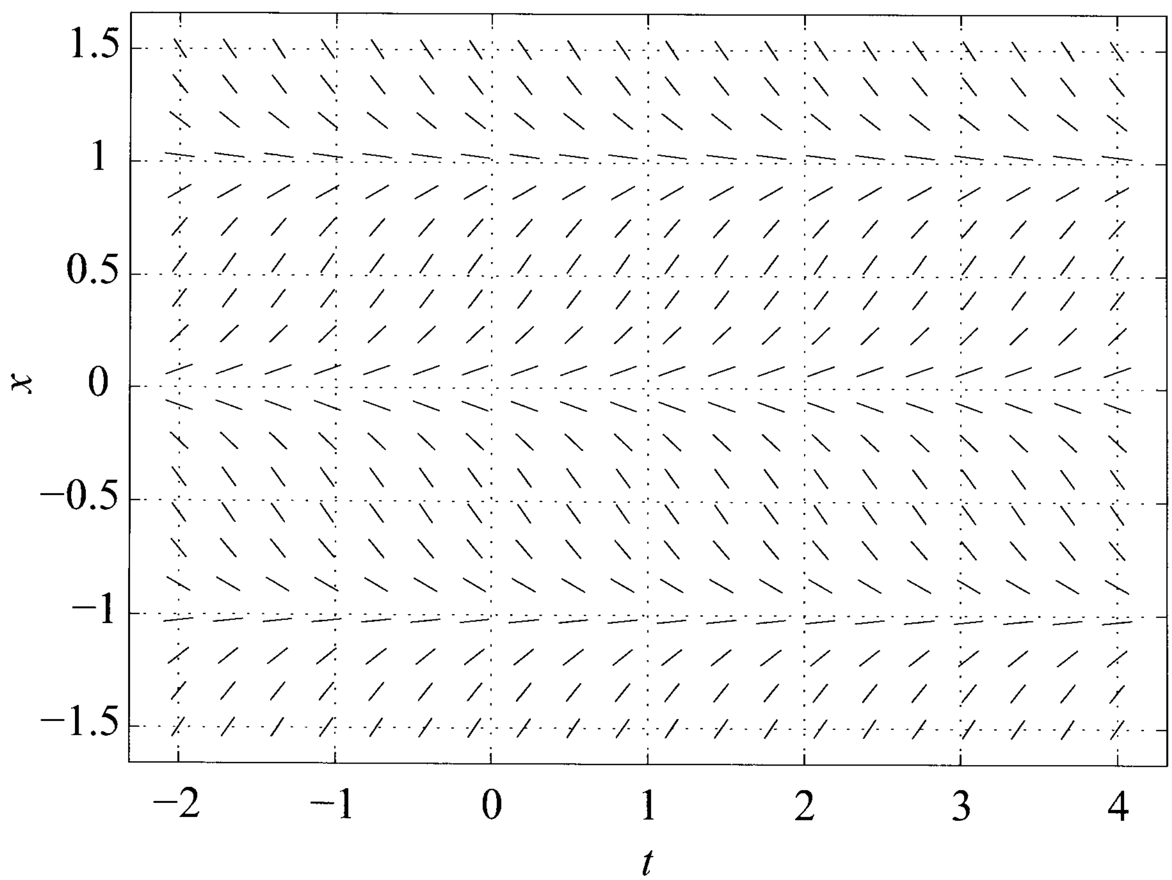
More generally, $\frac{dy}{dx} = g(x)$ with $g$ even, or $\frac{dy}{dx} = g(y)$ with $g$ even, are differential equations whose families of solutions are symmetric with respect to the origin.

(b) (5 points) Give an example of a differential equation whose family of solution curves is symmetric with respect to the $y$-axis, where $y$ is the dependent variable. Explain why you think the differential equation in question has this particular symmetry.

If now we set $\frac{dy}{dx} = x$, we see that the equation is unchanged if we change $x$ into $-x$. As a consequence, the family of solution curves of this differential equation is symmetric with respect to the $y$-axis.

More generally, $\frac{dy}{dx} = g(x)$ with $g$ odd has a family of solution curves which is invariant under reflection across the $y$-axis.

**Extra Credit (10 Points):**
Find a differential equation whose slope field "looks like" the one shown in the figure below. There is more than one possible solution to this problem. You may want to check your answer on the computer or on your calculator.




This slope field has equilibria at $x=-1$, $x=0$, and $x=1$. Both $x=-1$ and $x=1$ are stable, and $x=0$ is unstable.

For instance, setting $\frac{dx}{dt} = \sin(\pi x)$ works.