Sequences & series (continued)

4. Tests for convergence (continued)

Example 4: \[ \sum_{n=5}^{\infty} \frac{n \cdot 2^n}{3^n} \]

Let \( a_n = \frac{n \cdot 2^n}{3^n} \)

1. Does \( a_n \) go to 0 as \( n \to \infty \)?

\[ a_n = n \left( \frac{2}{3} \right)^n \]

\[ = n \cdot e^{n \ln \left( \frac{2}{3} \right)} \to 0 \quad \text{as} \quad n \to \infty \]

The exponential wins!

2. Ratio test: \[ \frac{a_{n+1}}{a_n} = \frac{(n+1) \cdot (2/3)^{n+1}}{n \cdot (2/3)^n} \]

\[ = \frac{n+1}{n} \cdot \frac{2}{3} \]

As \( n \to \infty \), \[ \frac{a_{n+1}}{a_n} \to \frac{2}{3} < 1 \]

So the series converges.

3. Could we have used the integral test?

\[ \int_1^{\infty} x \cdot \left( \frac{2}{3} \right)^x \, dx = \int_1^{\infty} x \cdot e^{x \ln \left( \frac{2}{3} \right)} \, dx \]

converges
Example 5: \[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^n}{n^2} \]

\[ \frac{2^n}{n^2} = \frac{e^n \ln(2)}{n^2} \rightarrow \infty \text{ as } n \rightarrow \infty \]

Since \[ \lim_{n \rightarrow \infty} \frac{(-1)^{n-1} 2^n}{n^2} \neq 0 \], the series diverges.

Could you have used the ratio test?

\[ \frac{a_{n+1}}{a_n} = \frac{(-1)^n}{(-1)^{n-1}} \cdot \frac{2^{n+1}}{2^n} \cdot \frac{n^2}{(n+1)^2} \]

\[ = -1 \cdot 2 \cdot \left( \frac{n}{n+1} \right)^2 \]

\[ \left| \frac{a_{n+1}}{a_n} \right| = 2 \left| \frac{n}{n+1} \right|^2 \rightarrow 2 \text{ as } n \rightarrow \infty \]

Since \( 2 > 1 \), the series diverges.