Linear differential equations with constant coefficients (continued)

Example 2: \( y'' + 6y' + 25y = 0 \) \( y(0) = 0 \); \( y'(0) = 1 \).

Characteristic equation: \( d^2 + 6d + 25 = 0 \)
\[ (d + 3)^2 - 9 + 25 = 0 \]
\[ i^2 16 = (i4)^2 \]
\[ d + 3 = \pm 4i \]
\[ d = -3 \pm 4i \]

Alternatively, use the quadratic formula:
\[
d = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
\[
= \frac{-6 \pm \sqrt{36 - 4 \cdot 25}}{2} = \frac{-6 \pm \sqrt{-8}}{2}
\]
\[
= \frac{-6 \pm 8i}{2} = -3 \pm 4i
\]

Since we have 2 roots, we have 2 linearly independent solutions:
\[ Y_1(x) = e^{(3+4i)x} \quad \text{and} \quad Y_2(x) = e^{(3-4i)x} \]

The general solution is of the form:
\[ y_h(x) = A e^{(3+4i)x} + B e^{(3-4i)x} \]

Since we want \( y_h \) to be real, \( A \) & \( B \) have to be complex.
What is $e^{(-3+4i)x} = e^{-3x} e^{4ix}$

\[ = e^{-3x} \left( \cos(4x) + i\sin(4x) \right) \]

Recall [Euler's formula]: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

To check (or to know where this comes from):

Start with the series expansion of $e^x$:

\[ e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \ldots \]

Do the algebra:

\[ (i^2) = -1 \]

\[ e^{i\theta} = \cos(\theta) + i\sin(\theta) \]

The series of $\cos(\theta)$ and $\sin(\theta)$ give:

- Unit circle

Now go back to the expression for $y_h(x) = A e^{(-3+4i)x} + B e^{(-3-4i)x}$:

\[ y_h(x) = A e^{-3x} \left( \cos(4x) + i\sin(4x) \right) + B e^{-3x} \left( \cos(-4x) + i\sin(-4x) \right) \]

\[ = A e^{-3x} \left( \cos(4x) + i\sin(4x) \right) + B e^{-3x} \left( \cos(-4x) - i\sin(-4x) \right) \]

Note that $e^{4ix}$ is the complex conjugate of $e^{-4ix}$. 
\[ y_h(x) = A e^{-3x} e^{4ix} + B e^{-3x} e^{-4ix} \]

Recall that we want \( y_h(x) \) to be real (and remember that \( A \) & \( B \) are complex constants). To say that \( y_h \) is real is the same thing as saying that it is equal to its complex conjugate:
\[ y_h(x) = \overline{y_h(x)} \]
i.e.
\[ A e^{-3x} e^{4ix} + B e^{-3x} e^{-4ix} = A e^{-3x} e^{4ix} + B e^{-3x} e^{-4ix} \]
\[ = A e^{-3x} e^{4ix} + B e^{-3x} e^{-4ix} \]
\[ = A e^{-3x} e^{-4ix} + B e^{-3x} e^{4ix} \]
\[ = \overline{A e^{-3x} e^{4ix} + B e^{-3x} e^{-4ix}} \]
(1) = (2) \( \iff \)
\[ A e^{-3x} e^{4ix} + B e^{-3x} e^{-4ix} \]
\[ = \overline{A e^{-3x} e^{4ix} + B e^{-3x} e^{-4ix}} \]
\[ = \overline{A e^{-3x} e^{4ix}} + \overline{B e^{-3x} e^{-4ix}} \]
in other words, the above amounts to imposing \( y_h(x) = \overline{y_h(x)} \), i.e. that \( y_h(x) \) is real.
i.e.
\[ 0 = e^{-3x} \left[ A e^{4ix} + B e^{-4ix} - \overline{A e^{-4ix}} - \overline{B e^{4ix}} \right] \]
\[ = e^{-3x} \left( (A - \overline{B}) e^{4ix} + (B - A) e^{-4ix} \right) \]
Note that the above must be true for all \( x \)'s. Since \( e^{-3x} \neq 0 \), we have
\[ (A - \overline{B}) e^{4ix} + (B - A) e^{-4ix} = 0 \]
i.e. a linear combination of \( e^{4ix} \) & \( e^{-4ix} \) equal to 0. Since \( e^{4ix} \) & \( e^{-4ix} \) are linearly independent (they're not proportional to one another), we get
\[ A = \overline{B} \text{ & } B = \overline{A} \] i.e. \( A \) & \( B \) are complex conjugates of one another.