Linear differential equations with constant coefficients (continued)

Why does the method of variation of parameters work?

Recall: Consider \( ay'' + by' + cy = 0 \)
Assume \( y_1(x) \) is a solution.
Look for another solution in the form \( y_2 = u(x)y_1(x) \)
Substitute in to get:
\[
v'(ay_1) + v(2ay_1' + by_1) = 0
\]

Now assume that \( y_1 \) is an exponential, i.e., \( y_1(x) = e^{ax} \).

Then we have \( y_1' = ae^{ax} \) and the ODE for \( v \) is
\[
v'a e^{ax} + v(2ae^{ax} + be^{ax}) = 0
\]
i.e.,
\[
v'a e^{ax} + v(2ad + b)e^{ax} = 0
\]
Since \( e^{ax} \neq 0 \), then
\[
a v' + v(2ad + b) = 0.
\]
Recall that \( a \) solves \( ad^2 + bd + c = 0 \)
but in the special case where there is only 1 root, i.e., \( b^2 - 4ac = 0 \).
In this case, \[ d = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a}, \]
i.e. \[ 2ad + b = 0. \]

The ODE for \( v \) is then \( a v' = 0 \), i.e. (since \( a \neq 0 \)), \( v' = 0. \)

Then \( v = \text{constant} = C \) and \( u = C_1 x + C_2 \) (since \( u' = v \)).

Recall that we are looking for \( 1 \, y_2(x) \), so we can pick simple values of \( C_1, 2C_2 \).

- Can I pick \( C_1 = 0 \)? No because then \( u = C_2 \) and \( y_2(x) = C_2 \, y_1(x) \), i.e. is not linearly independent from \( y_1 \).

- Pick \( C_2 = 0 \) and \( C_1 = 1 \).

Then \( y_1(x) = x \, y_1(x) \).

The general solution to \( ay'' + by' + cy = 0 \) is, in this case, given by

\[ y(x) = C_1 \, y_1(x) + C_2 \, y_2(x) = C_1 \, e^{ax} + C_2 \, x \, e^{2x}. \]

Remark: The method of reduction of order works even if \( a, b \) and \( c \) are not constant functions \( \Rightarrow \) it is extremely general for linear equations.