Example 3: \( y'' + 6y' + 25y = \cos(4x) + x \)

Assume we have a particular solution \( y_1 \) to
\[ y'' + 6y' + 25y = \cos(4x) \]

and a particular solution \( y_2 \) to
\[ y'' + 6y' + 25y = x \]

Then
\[
\begin{align*}
y_{1''} + 6y_{1'} + 25y_1 &= \cos(4x) \\
y_{2''} + 6y_{2'} + 25y_2 &= x
\end{align*}
\]

Add the 2 equations:
\[ y_{1''} + y_{2''} + 6y_{1'} + 6y_{2'} + 25y_1 + 25y_2 = \cos(4x) + x \]

i.e. \((y_1 + y_2)'' + 6(y_1 + y_2) + 25(y_1 + y_2) = \cos(4x) + x\)

Set \( y_p = y_1 + y_2 \). Then
\[ y_p'' + 6y_p' + 25y_p = \cos(4x) + x \]

i.e. \( y_p \) is a particular solution to
\[ y'' + 6y' + 25y = \cos(4x) + x \]
Thus, a particular solution to
\[ y'' + 6y' + 25y = \cos(4x) + x \]
will be of the form
\[ y_p = \frac{A \cos(4x) + B \sin(4x)}{\alpha = 0 \quad \beta = 4} + \frac{C x + D}{\alpha = 0 \quad \beta = 0} \]
\[ \text{4i is not a root of } d^2 + 6d + 25 = 0 \]
\[ \text{0 is not a root of } d^2 + 6d + 25 = 0 \]

We've in fact already found a particular solution to \[ y'' + 6y' + 25y = \cos(4x) \]. This was \[ y_{p1} = \frac{1}{72} \left( \cos(4x) + \frac{8}{3} \sin(4x) \right) \].

We look for \( y_{p2} \), which is a particular solution to \( y'' + 6y' + 25y = x \), in the form
\[ y_{p2} = C x + D \]

\[ y_{p2}' = C \quad y_{p2}'' = 0 \]
\[ y_{p2}'' + 6y_{p2}' + 25y_{p2} = x \]
\[ (\alpha = 0 \quad \beta = 6) \]
\[ 0 + 6C + 25(Cx + D) = x \]
\[ 25C x + (6C + 25D) = x \]
\[ \{ \quad 25C = 1 \quad (\alpha = 0 \quad \beta = 6) \}
\[ 6C + 25D = 0 \]
\[ D = \frac{1}{25} (-6C) = \frac{-6}{625} \]
\[ \Rightarrow C = \frac{1}{25} \]

So \[ y_{p2} = \frac{1}{25} \left( x - \frac{6}{25} \right) \].
A particular solution is

\[ y' + 6y' + 25y = \cos(4x) + x \]

is

\[ y_p = \frac{1}{72} \left( \cos(4x) + \frac{8}{3} \sin(4x) \right) + \frac{1}{25} \left( x - \frac{6}{25} \right) \]