Improper integrals
There are two types of improper integrals, those for which one of the limits of integration is infinite, and those whose integrand diverges at some point in the interval of integration.

The improper integral \( \int_a^\infty f(x) \, dx \), where \( f \) is continuous on \([a, \infty)\), is defined as

\[
\int_a^\infty f(x) \, dx = \lim_{b \to +\infty} \int_a^b f(x) \, dx.
\]

If the limit exists and is finite, then one says that the integral converges. If not, one says that the integral diverges.

Remarks:

1. The definition of \( \int_{-\infty}^c f(x) \, dx \) is similar.

2. If \( \lim_{x \to +\infty} f(x) = a, \ a \neq 0 \), then \( \int_a^\infty f(x) \, dx \) cannot converge.
Example 1: Does $\int_1^\infty e^x \, dx$ converge?

1. Yes
2. No

Example 2: Does $\int_1^\infty \frac{1}{x^3} \, dx$ converge?

1. Yes
2. No

Example 3: When does $\int_1^\infty \frac{1}{x^p} \, dx$ converge?

Answer:

- If $p > 1$, then $\int_1^\infty \frac{1}{x^p} \, dx$ converges and is equal to $\frac{1}{p-1}$.
- If $p \leq 1$, then $\int_1^\infty \frac{1}{x^p} \, dx$ diverges.
We define \(\int_{-\infty}^{+\infty} f(x) \, dx\), where \(f\) is continuous on \(\mathbb{R}\), as
\[
\int_{-\infty}^{+\infty} f(x) \, dx = \int_{-\infty}^{a} f(x) \, dx + \int_{a}^{+\infty} f(x) \, dx, \quad a \in \mathbb{R}.
\]

As a consequence, for \(\int_{-\infty}^{+\infty} f(x) \, dx\) to converge, we need convergence of both \(\int_{-\infty}^{a} f(x) \, dx\) and \(\int_{a}^{+\infty} f(x) \, dx\).

Example: Can you find a function \(f(x)\) such that \(\int_{-\infty}^{0} f(x) \, dx\) converges but \(\int_{0}^{+\infty} f(x) \, dx\) diverges?
Assume that the function $f$ is continuous except at $x = a$, $a \in \mathbb{R}$, where it diverges.

Then, integrals of the form $\int_{a}^{c} f(x) \, dx$, $\int_{b}^{a} f(x) \, dx$, or $\int_{b}^{c} f(x) \, dx$ with $a \in (b, c)$, are all improper integrals.

As before, we have the following definition

$$\int_{a}^{c} f(x) \, dx = \lim_{z \to a^{+}} \int_{z}^{c} f(x) \, dx,$$

and similarly for $\int_{b}^{a} f(x) \, dx$.

If the above limit exits and is finite, then the corresponding integral converges. If not, it diverges.
Example 1: Does \( \int_0^2 \frac{dx}{\sqrt{4 - x^2}} \) converge? If so, find its value.

For \( a \in [b, c] \) we define

\[
\int_b^c f(x) \, dx = \int_b^a f(x) \, dx + \int_a^c f(x) \, dx,
\]

and both integrals have to converge for \( \int_b^c f(x) \, dx \) to converge.

Example 2: Does \( \int_{-1}^1 \frac{dx}{x^2} \) converge?

1. Yes
2. No

If the answer is yes, find its value.
Example 3: For which values of \( p \) does \( \int_0^1 \frac{dx}{x^p} \) converge?

Answer:

- If \( p \geq 1 \), then \( \int_0^1 \frac{dx}{x^p} \) diverges.
- If \( p < 1 \), then \( \int_0^1 \frac{dx}{x^p} \) converges, and is equal to \( \frac{1}{1 - p} \).

Example 4: Does \( \int_{-1}^1 \frac{du}{u} \) converge?

1. Yes
2. No

Example 5: Without integrating, can you decide whether \( \int_0^2 \frac{dx}{\sqrt{4 - x^2}} \) converge?
More on the notion of convergence

- Does \( \int_{0}^{\infty} \frac{x}{e^{x}} \, dx \) converge?
  1. Yes
  2. No

- Use your calculator and Simpson’s rule to approximate this integral. In other words, evaluate \( \int_{a}^{b} x e^{-x} \, dx \), for \( b = 10, 15, 20, 30, 40 \). What do you observe?

- Similarly, for \( \int_{0}^{2} \frac{dx}{\sqrt{4 - x^2}} \), use your calculator to evaluate \( \int_{0}^{b} \frac{dx}{\sqrt{4 - x^2}} \), for \( b = 1.9, 1.99, 1.999 \). What do you observe?
Comparison of improper integrals

- The methods discussed below are useful when you want to know **whether and improper integral converges** rather than its particular exact value.

- Also, if you **know in advance** that an integral diverges, then there is no need to try to evaluate it.

**Comparison theorems**

- If $0 \leq f(x) \leq g(x)$ and $\int_a^\infty g(x) \, dx$ converges, so does $\int_a^\infty f(x) \, dx$.

- If $0 \leq g(x) \leq f(x)$ and $\int_a^\infty g(x) \, dx$ diverges, so does $\int_a^\infty f(x) \, dx$. 
Example 1: Does $\int_{1}^{\infty} \frac{1}{1 + x^3} \, dx$ converge?

1. Yes
2. No

Example 2: Does $\int_{4}^{\infty} \frac{3 + \sin(\alpha)}{\alpha} \, d\alpha$ converge?

1. Yes
2. No

Example 3: Does $\int_{1}^{\infty} \frac{x^2 + 4}{x^4 + 3x^2 + 11} \, dx$ converge?

1. Yes
2. No