Sequences and series
A sequence is an infinite list of numbers, $s_1, s_2, \ldots, s_n, \ldots$, indexed by integers.

Example 1: Find the first five terms of $s_n = (-1)^n \left(\frac{1}{3}\right)^n$, $n \geq 1$.

Example 2: Find a formula for $s_n$, $n \geq 1$, given that its first five terms are $0, 2, 6, 14, 30$.

Some sequences are defined recursively. For instance, $s_n = 2s_{n-1} + 3$, $n > 1$, with $s_1 = 1$.

If $\lim_{n \to \infty} s_n = L$, where $L$ is a number, we say that the sequence $(s_n)$ converges to $L$. If such a limit does not exist or if $L = \pm \infty$, one says that the sequence diverges.
Example 3: Does the sequence \( \left( \frac{2^n}{5^n} \right) \) converge?

1. Yes
2. No

Example 4: Does the sequence \( \left( \frac{n}{2} + \frac{5}{n} \right) \) converge?

1. Yes
2. No

Example 5: Does the sequence \( \left( \frac{\sin(2n)}{n} \right) \) converge?

Remarks:

1. A convergent sequence is bounded, i.e. one can find two numbers \( M \) and \( N \) such that \( M < s_n < N \), for all \( n \)'s.
2. If a sequence is bounded and monotone, then it converges.
A series is a pair of sequences, \((S_n)\) and \((u_n)\) such that

\[ S_n = \sum_{k=1}^{n} u_k. \]

A geometric series is of the form

\[ S_n = a + ax + ax^2 + ax^3 + \cdots + ax^{n-1}, \quad u_k = ax^{k-1} \]

One can show that if \(x \neq 1\), \(S_n = a \frac{1 - x^n}{1 - x}\).

Example 1: Is \(2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\) geometric?

1. Yes
2. No
Example 2: Is \(1 + x + 2x^2 + 3x^3 + 4x^4 + \ldots\) geometric?

1. Yes
2. No

Example 3: Find \(3(0.1) + 3(0.1)^2 + 3(0.1)^3 + \cdots + 3(0.1)^{10}\).

Example 4: Assume \(\sum_{k=1}^{n} u_k\) is geometric and that \(u_k \neq 0\) for all \(k\)'s. Is \(\sum_{k=1}^{n} \frac{1}{u_k}\) geometric as well?

1. Yes
2. No
Recall that we defined the partial sum $S_n$ as $S_n = \sum_{k=1}^{n} u_k$.

If $S_n$ converges and $\lim_{n \to \infty} S_n = S$, then one says that the infinite series $\sum_{k=1}^{\infty} u_k$ converges, and that its sum is equal to $S$. One writes $\sum_{k=1}^{\infty} u_k = S$.

If $\lim_{n \to \infty} S_n$ does not exist or is infinite, one says that the series $\sum_{k=1}^{\infty} u_k$ diverges.
Recall that an infinite geometric series is written as an infinite sum of the form

\[ a + ax + ax^2 + \cdots + ax^n + \ldots \quad a, \ x \neq 0 \]

If \( |x| < 1 \), the above series converges to \( \frac{a}{1 - x} \).

If \( |x| > 1 \), the series diverges.

If \( x = 1 \), \( \sum_{k=1}^{n} a = na \) and the series diverges.

If \( x = -1 \), \( \sum_{k=1}^{n} a = a - a + a - a + \ldots \), and the series diverges.
More on convergence of series

- If \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) converge, and if \( k \) is a constant, then
  \[
  \sum_{n=1}^{\infty} (a_n + b_n)
  \]
  converges to \( \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \).

- \( \sum_{n=1}^{\infty} k a_n \) converges to \( k \sum_{n=1}^{\infty} a_n \).

- Changing a finite number of terms in a series does not change its convergence properties.

- If \( \lim_{n \to \infty} a_n \) is not zero, or if it does not exist, then \( \sum_{n=1}^{\infty} a_n \) diverges.

- If \( \sum_{n=1}^{\infty} a_n \) diverges, so does \( \sum_{n=1}^{\infty} k a_n \), for \( k \neq 0 \).
Comparison of series with integrals

- There is a direct correspondence between convergence of series and convergence of integrals. This is called the integral convergence test.

- Assume that $a_n = f(n)$, where $f$ is a continuous, decreasing function that remains positive for $x > b$. Then,
  - If $\int_b^\infty f(x) \, dx$ converges, so does $\sum_{n=1}^n a_n$.
  - If $\int_b^\infty f(x) \, dx$ diverges, so does $\sum_{n=1}^n a_n$.

- This can be understood by writing the series as a left or right sum approximation of the improper integral.
Example 1: Does $\sum_{n=1}^{\infty} \frac{1}{e^n}$ converge?  
1. Yes  
2. No  

Example 2: Does $\sum_{n=1}^{\infty} \frac{4}{(2n + 3)^3}$ converge?  
1. Yes  
2. No  

Example 3: Does $\sum_{n=1}^{\infty} \frac{1}{1 + n}$ converge?  
1. Yes  
2. No
Tests for convergence

- **Theorem 1: The Comparison Test**
  Suppose $0 \leq a_n \leq b_n$ for all $n$’s. Then,
  1. If $\sum_{n=1}^{\infty} b_n$ converges, so does $\sum_{n=1}^{\infty} a_n$.
  2. If $\sum_{n=1}^{\infty} a_n$ diverges, so does $\sum_{n=1}^{\infty} b_n$.

- **Theorem 2:** Suppose $a_n > 0$ and $b_n > 0$ for all $n$’s. If
  \[
  \lim_{n \to \infty} \frac{a_n}{b_n} = c, \quad c > 0,
  \]
  then the two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.
Definitions:

1. The series \( \sum_{n=1}^{\infty} a_n \) is absolutely convergent if \( \sum_{n=1}^{\infty} |a_n| \) converges.

2. The series \( \sum_{n=1}^{\infty} a_n \) is conditionally convergent if \( \sum_{n=1}^{\infty} |a_n| \) diverges but \( \sum_{n=1}^{\infty} a_n \) converges.

Theorem 3: Absolute convergence implies convergence

If \( \sum_{n=1}^{\infty} |a_n| \) converges, so does \( \sum_{n=1}^{\infty} a_n \).
Theorem 4: The Ratio Test

Consider the series \( \sum_{n=1}^{\infty} a_n \) and suppose that

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L.
\]

Then,

1. If \( L < 1 \), then the series \( \sum_{n=1}^{\infty} a_n \) converges.
2. If \( L > 1 \), then the series \( \sum_{n=1}^{\infty} a_n \) diverges.
3. If \( L = 1 \), more information is needed to conclude.
Theorem 5: The Alternating Series Test
Consider the series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ and suppose that

$$\lim_{n \to \infty} a_n = 0,$$

and

$$0 < a_{n+1} < a_n.$$

Then, the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges.

Theorem 6: Error bounds for alternating series
Assume that the conditions of Theorem 5 are satisfied, and let

$$S = \sum_{n=1}^{\infty}(-1)^{n-1}a_n, \quad S_n = \sum_{k=1}^{n}(-1)^{k-1}a_k.$$ 

Then, $|S - S_n| < a_{n+1}$ for all $n$'s.
Example

Consider the series \[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}. \]

Can we use the ratio test to conclude convergence?
1. Yes
2. No

Can we use the alternating series test instead?
1. Yes
2. No

The series is
1. Conditionally convergent
2. Absolutely convergent

How many terms should we include in the partial sum \( S_n \) to guarantee that \( S_n \) is an approximation of \( S \) correct to 3 decimal places?
More examples

- Does the series \( \sum_{n=0}^{\infty} \frac{2}{\sqrt{2 + n}} \) converge?
  1. Yes
  2. No

- Does the series \( \sum_{n=1}^{\infty} \frac{n + 2^n}{n 2^n} \) converge?
  1. Yes
  2. No

- Does the series \( \sum_{n=5}^{\infty} \frac{n 2^n}{3^n} \) converge?
  1. Yes
  2. No

- Does the series \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^n}{n^2} \) converge?
  1. Yes
  2. No
A power series is a series of the form
\[ \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots + c_n (x-a)^n + \ldots \]

Abel’s lemma: If the power series converges for \(|x - a| = R_0\), then it converges absolutely for all \(x\)’s such that \(|x - a| < R_0\).

As a consequence, one can define its radius of convergence \(R\):
- If the series only converges for \(x = a\), one says that \(R = 0\).
- If the series converges for all \(x\)’s, one says that \(R = \infty\).
- Otherwise, \(R\) is the largest number such that the series converges for \(|x - a| < R\).

The interval of convergence of the power series is \((a - R, a + R)\) plus any end point where the series converges.
Example 1: What are the radius of convergence and the interval of convergence of the series \( \sum_{n=1}^{\infty} \frac{(x - 2)^n}{n} \)?

To find the radius of convergence of a power series, use the ratio test.

Example 2: Find the interval of convergence of the power series \( \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \).

Example 3: Find the interval of convergence of the power series \( \sum_{n=1}^{\infty} (-1)^n \frac{(x - 3)^n}{2^n n^2} \).
Which of the following sequences are monotone and bounded?

I. \( s_n = 10 - \frac{1}{n} \)  
II. \( s_n = \frac{10n + 1}{n} \)  
III. \( s_n = \cos(n) \)  
IV. \( s_n = \ln(n) \)

1. I only
2. I and II
3. II and IV
4. I, II, and III

Which of the following sequences converge?

I. \( s_n = \frac{\cos(n)}{n} \)  
II. \( s_n = \cos \left( \frac{1}{n} \right) \)  
III. \( s_n = n e^{-n} \)  
IV. \( s_n = \frac{\sin(1/n)}{1/n} \)

1. I and III
2. I, II, and III
3. II and IV
4. All of them
Which of the following geometric series converge?

I. $20 - 10 + 5 - 2.5 + \ldots$

II. $1 - 1.1 + 1.21 - 1.331 + \ldots$

III. $1 + 1.1 + 1.21 + 1.331 + \ldots$

IV. $1 + y^2 + y^4 + y^6 \ldots \quad -1 < y < 1$

1. I only
2. IV only
3. I and IV
4. II and IV
5. None of them
Which of the following statements are true?

(a) If \( \lim_{n \to \infty} a_n = 0 \), then \( \sum_{n=0}^{\infty} a_n \) converges.

(b) If \( \lim_{n \to \infty} a_n \neq 0 \), then \( \sum_{n=0}^{\infty} a_n \) diverges.

(c) If \( \sum_{n=0}^{\infty} a_n \) converges, then \( \lim_{n \to \infty} a_n = 0 \).

1. (a) and (c) only
2. (a), (b), and (c)
3. (a) and (b) only
4. (b) and (c) only
Suppose that the series \( \sum_{n=5}^{\infty} c_n x^n \) converges for \( x = -5 \) and diverges for \( x = 8 \). Which of the following statements are certainly true?

(a) The series converges when \( x = 12 \).
(b) The series converges when \( x = 1 \).
(c) The series diverges when \( x = 3 \).
(d) The series diverges when \( x = 7 \).

1. (a) and (c) only
2. (b) only
3. (a), (c) and (d) only
4. All of them