Second Order ODEs - Review Problems

April 13, 2009

1. Find a particular solution to
\[ x^5 y'' + 3xy' + 7y = 14. \]

2. Consider the following equation
\[ \frac{d^2x}{dt^2} + 25x = 16 \cos(3t). \]
   
   (a) Find the general solution of this equation.
   
   (b) Show that the particular solution satisfying \( x(0) = 0 \) and \( \dot{x}(0) = 0 \) is \( x_p(t) = \cos(3t) - \cos(5t) \).
   
   (c) Using the identity \( \cos(3t) - \cos(5t) = 2 \sin(4t) \sin(t) \), sketch the graph of the particular solution found in (b) for \( 0 \leq t \leq 2\pi \).

3. Solve the differential equation
\[ y'' + 6y' + 9y = \frac{1}{x} e^{-3x}. \]

4. Answer the following questions about the solution \( x(t) \) to the damped equation
\[ m\ddot{x} + k\dot{x} + hx = 0, \]
where \( m, k \) and \( h \) are positive constants.
   
   (a) If \( m = 2 \), how should \( h \) and \( k \) be related so that the non-zero solutions are oscillatory?
   
   (b) If \( h = k = 1 \), how should the mass \( m \) be chosen so that all non-zero solutions will oscillate?
   
   (c) If \( m = h = 1 \), how should \( k \) be chosen so that \( x(t) \) is oscillatory?
5. Solve the differential equation

\[ y'' - 3y' + 2y = -\frac{e^{2x}}{e^x + 1}. \]

**Hint:** \( 1/(e^t + 1) \) can be integrated by noticing that it is equal to \( e^{-t}/(1 + e^{-t}) \).

6. (a) Find a particular solution to the equation

\[ \frac{d^2 x}{dt^2} + 22x = \cos(\gamma t), \]

where \( \gamma > 0 \). Note that your answer should depend on \( \gamma \). For what value of \( \gamma \) is there resonance?

(b) Find a particular solution \( x_p(t) \) to

\[ \frac{d^2 x}{dt^2} + 6\frac{dx}{dt} + 22x = \cos(\gamma t). \]

What is the value of \( x_p \) at \( t = \pi/(2\gamma) \)? Sketch the behavior of this value as a function of \( \gamma \). What happens when \( \gamma = \sqrt{22} \)?

7. Solve the differential equation

\[ y'' + 2y' + y = \frac{2}{x^2}e^{-x}. \]