Review of Complex Numbers

Definitions, Algebra of complex numbers, Polar coordinates
1. Complex numbers

- Complex numbers are of the form

\[ z = x + iy, \quad x, y \in \mathbb{R}, \quad i^2 = -1. \]

- In the above definition, \( x \) is the real part of \( z \) and \( y \) is the imaginary part of \( z \).

- The complex number \( z = x + iy \) may be represented in the complex plane as the point with cartesian coordinates \((x, y)\).
The complex conjugate of $z = x + iy$ is defined as

$$\bar{z} = x - iy.$$

As a consequence of the above definition, we have

$$\Re(z) = \frac{z + \bar{z}}{2}, \quad \Im(z) = \frac{z - \bar{z}}{2i}, \quad z\bar{z} = x^2 + y^2. \quad (1)$$

If $z_1$ and $z_2$ are two complex numbers, then

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}, \quad \overline{z_1z_2} = \overline{z_1} \overline{z_2}. \quad (2)$$
Modulus of a complex number

- The **absolute value or modulus** of $z = x + iy$ is
  
  $$|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}.$$  

  It is a **positive number**.

- **Examples**: Evaluate the following
  
  - $|i|$
  - $|2 - 3i|$
2. Algebra of complex numbers

- You should use the same rules of algebra as for real numbers, but remember that $i^2 = -1$.

**Examples:**
- Find powers of $i$ and $1/i$.
- Assume $z_1 = 2 + 3i$ and $z_2 = -1 - 7i$. Calculate $z_1z_2$ and $(z_1 + z_2)^2$.

- Get used to writing a complex number in the form

$$z = \text{(real part)} + i \text{ (imaginary part)},$$

no matter how complicated this expression might be.
Remember that multiplying a complex number by its complex conjugate gives a real number.

**Examples:** Assume $z_1 = 2 + 3i$ and $z_2 = -1 - 7i$.

- Find $z_1/z_2$.
- Find $\overline{z}_1/\overline{z}_2$.
- Find $\Im(1/\overline{z}_1^3)$.
3. Polar coordinates form of complex numbers

- In polar coordinates,
  \[ x = r \cos(\theta), \quad y = r \sin(\theta), \]
  where
  \[ r = \sqrt{x^2 + y^2} = |z|. \]

- The angle \( \theta \) is called the argument of \( z \). It is defined for all \( z \neq 0 \), and is given by

\[
\text{arg}(z) = \theta = \begin{cases} 
\arctan \left( \frac{y}{x} \right) & \text{if } x \geq 0 \\
\arctan \left( \frac{y}{x} \right) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\
\arctan \left( \frac{y}{x} \right) - \pi & \text{if } x < 0 \text{ and } y < 0 
\end{cases} \pm 2n\pi.
\]
Because \( \arg(z) \) is multi-valued, it is convenient to agree on a particular choice of \( \arg(z) \), in order to have a single-valued function.

The principal value of \( \arg(z) \), \( \text{Arg}(z) \), is such that

\[
\tan (\text{Arg}(z)) = \frac{y}{x}, \quad \text{with} \quad -\pi < \text{Arg}(z) \leq \pi.
\]

Note that \( \text{Arg}(z) \) jumps by \( -2\pi \) when one crosses the negative real axis from above.
Principal value $\text{Arg}(z)$ (continued)

- **Examples:**
  - Find the principal value of the argument of $z = 1 - i$.
  - Find the principal value of the argument of $z = -10$. 

![Graph showing the complex plane with points labeled 0, 1, and 1 along the real and imaginary axes.](image)
You need to be able to go back and forth between the polar and cartesian representations of a complex number.

\[ z = x + iy = |z| \cos(\theta) + i|z| \sin(\theta). \]

In particular, you need to know the values of the sine and cosine of multiples of \( \pi/6 \) and \( \pi/4 \).

- Convert \( \cos \left( \frac{\pi}{6} \right) + i \sin \left( \frac{\pi}{6} \right) \) to cartesian coordinates.

- What is the cartesian form of the complex number such that \( |z| = 3 \) and \( \text{Arg}(z) = \pi/4 \)?
Euler’s formula

- Euler’s formula reads

\[ \exp(i\theta) = \cos(\theta) + i\sin(\theta), \quad \theta \in \mathbb{R}. \]

- As a consequence, every complex number \( z \neq 0 \) can be written as

\[ z = |z| (\cos(\theta) + i\sin(\theta)) = |z| \exp(i\theta). \]

- This formula is extremely useful for calculating powers and roots of complex numbers, or for multiplying and dividing complex numbers in polar form.
Integer powers of a complex number

To find the $n$-th power of a complex number $z \neq 0$, proceed as follows

1. Write $z$ in exponential form, $z = |z| \exp (i\theta)$.
2. Then take the $n$-th power of each side of the above equation
   \[ z^n = |z|^n \exp (i n \theta) = |z|^n (\cos(n \theta) + i \sin(n \theta)) . \]
3. In particular, if $z$ is on the unit circle ($|z| = 1$), we have
   \[ (\cos(\theta) + i \sin(\theta))^n = \cos(n \theta) + i \sin(n \theta) . \]

This is De Moivre’s formula.
Examples of application:

- Trigonometric formulas

\[
\begin{align*}
\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta), \\
\sin(2\theta) &= 2\sin(\theta)\cos(\theta).
\end{align*}
\]

- Find \(\cos(3\theta)\) and \(\sin(3\theta)\) in terms of \(\cos(\theta)\) and \(\sin(\theta)\).
Product of two complex numbers

- The product of $z_1 = r_1 \exp(i\theta_1)$ and $z_2 = r_2 \exp(i\theta_2)$ is
  \[
  z_1 z_2 = (r_1 \exp(i\theta_1)) \cdot (r_2 \exp(i\theta_2)) = r_1 r_2 \exp(i(\theta_1 + \theta_2)).
  \] (4)

- As a consequence,
  \[
  \arg(z_1 z_2) = \arg(z_1) + \arg(z_2), \quad |z_1 z_2| = |z_1| |z_2|.
  \]

- We can use Equation (4) to show that
  \[
  \cos(\theta_1 + \theta_2) = \cos(\theta_1) \cos(\theta_2) - \sin(\theta_1) \sin(\theta_2),
  \] (5)
  \[
  \sin(\theta_1 + \theta_2) = \sin(\theta_1) \cos(\theta_2) + \cos(\theta_1) \sin(\theta_2).
  \]
Similarly, the ratio $z_1/z_2$ of 2 complex numbers is given by

$$\frac{z_1}{z_2} = \frac{r_1 \exp(i \theta_1)}{r_2 \exp(i \theta_2)} = \frac{r_1}{r_2} \exp(i(\theta_1 - \theta_2)).$$

As a consequence,

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2), \quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$

**Example:** Assume $z_1 = 2 + 3i$ and $z_2 = -1 - 7i$. Find $\left|\frac{z_1}{z_2}\right|$. 
**Roots of a complex number**

To find the \( n \)-th roots of a complex number \( z \neq 0 \), proceed as follows (see the MIT applet for a graphic representation)

1. Write \( z \) in exponential form, \( z = r \exp(i(\theta + 2p\pi)) \), with \( r = |z| \) and \( p \in \mathbb{Z} \).
2. Then take the \( n \)-th root (or the \( 1/n \)-th power)

\[
\sqrt[n]{z} = z^{1/n} = r^{1/n} \exp \left( i \frac{\theta + 2p\pi}{n} \right) = \sqrt[n]{r} \exp \left( i \frac{\theta + 2p\pi}{n} \right).
\]

3. There are thus \( n \) roots of \( z \), given by

\[
z_k = \sqrt[n]{r} \left( \cos \left( \frac{\theta + 2k\pi}{n} \right) + i \sin \left( \frac{\theta + 2k\pi}{n} \right) \right), \quad k = 0, \ldots, n-1.
\]
The principal value of $\sqrt[n]{z}$ is the $n$-th root of $z$ obtained by taking $\theta = \text{Arg}(z)$ and $k = 0$.

The $n$-th roots of unity are given by

$$\sqrt[n]{1} = \cos \left( \frac{2k\pi}{n} \right) + i \sin \left( \frac{2k\pi}{n} \right) = \omega^k, \quad k = 0, \ldots, n-1$$

where $\omega = \cos(2\pi/n) + i \sin(2\pi/n)$.

In particular, if $w_1$ is any $n$-th root of $z \neq 0$, then the $n$-th roots of $z$ are given by

$$w_1, \ w_1 \omega, \ w_1 \omega^2, \ldots, \ w_1 \omega^{n-1}.$$
**Examples:**

- Find the three cubic roots of 1.

- Find the four values of $\sqrt[4]{i}$.

- Give a representation in the complex plane of the principal value of the eighth root of $z = -3 + 4i$. 


If $z_1$ and $z_2$ are two complex numbers, then

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$ 

This is called the triangle inequality. Geometrically, it says that the length of any side of a triangle cannot be larger than the sum of the lengths of the other two sides.

More generally, if $z_1, z_2, \ldots, z_n$ are $n$ complex numbers, then

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|.$$
Check your knowledge

Are you comfortable with the following algebraic manipulations?

- Addition, subtraction, multiplication, and division of complex numbers;
- Finding the real and imaginary parts and the modulus of combinations of complex numbers;
- Simplifying expressions that involve $z$ and its complex conjugate.
Check your knowledge (continued)

Are you comfortable with the following algebraic manipulations?

- Going back and forth between polar and Cartesian representations of a complex number;
- Finding the argument, roots, logarithm, exponential, powers of a complex number.
- Given a complex number $z$, can you evaluate the principal value at $z$ of each of the multi-valued functions we discussed in class?