Partial Differential Equations

Introduction

1. The derivative of a function of one variable: Review

- If f is a function of the variable $x \in \mathbb{R}$, the derivative of f at x, $f'(x)$, is defined as

$$f'(x) = \lim_{\epsilon \longrightarrow 0} \frac{f(x+\epsilon) - f(x)}{\epsilon}.$$

  If this limit exists, one says that f has a derivative or is differentiable at x.

- If you need to review the concept of derivative of a function of one variable, you may for instance want to consult the corresponding Wikipedia article.

The derivative of a function of one variable (continued)

- Since the derivative is the limit, as ϵ goes to zero, of the slope of the secant to $y = f(x)$ at $(x, f(x))$ and $(x + \epsilon, f(x + \epsilon))$, $f'(x)$ measures the slope of the graph of f at the point x.

- The MIT *derivative and tangent line* applet illustrates this concept. Experiment with this applet until you feel comfortable with both the geometric and analytic descriptions of the derivative.

Check your understanding

1. Using the definition of $f'(x)$ as a limit, show that if $f(x) = 3x^2$, then $f'(x) = 6x$, for every $x \in \mathbb{R}$.
2. What does the sign of the derivative tell you about the function? Why?
3. What does it mean for the function f if its derivative is equal to zero at every point? Explain.
4. What does it mean for the graph of the function f near the point $(x, f(x))$, if $f'(x) = 0$? What if $f''(x) = 0$ as well? Why?

2. Partial derivatives: Introduction

- Consider now a function of two variables, $f(x, y)$, where x and y are in $\mathbb{R}$. If we fix the variable y to say $y = y_0$, we are left with a function of one variable, $g(x) = f(x, y_0)$.

- The partial derivative of f with respect to x at (x, y_0) is the derivative of the function g with respect to x. In other words,

$$\frac{\partial f}{\partial x}(x, y_0) = \lim_{\epsilon \longrightarrow 0} \frac{f(x+\epsilon, y_0) - f(x, y_0)}{\epsilon}.$$

Partial derivatives (continued)

- The partial derivative of f with respect to y is defined in a similar fashion.
- Since the partial derivative can be understood as the derivative of a function of one variable, all of the rules of differentiation that you learned in Calculus I apply.
- Partial derivatives of higher order are defined in a way similar to higher order derivatives of a function of one variable.

Check your understanding

1. Use the definition (in terms of a limit) of the partial derivative to find $\dfrac{\partial f}{\partial x}$ as a function of x and y, for $f(x, y) = 3x^2 + x\,y$.
2. Repeat the above calculation using standard rules of differentiation.
3. Use the rules of differentiation to calculate the following partial derivatives
   (a) $\dfrac{\partial f}{\partial y}(x, y)$, where $f(x, y) = \cos(x\,y)$.
   (b) $\dfrac{\partial f}{\partial x}(3, 5)$, where $f(x, y) = x^2 y^4 \exp(3x + y)$.
   (c) $\dfrac{\partial f}{\partial y}(x, y)$, where $f(x, y) = g(z)$ and $z = x\,y$.
   (d) $\dfrac{\partial^2 f}{\partial y^2}(x, y)$, where $f(x, y) = \cos(x\,y)$.

3. Partial differential equations: Introduction

- A partial differential equation (PDE) is an equation which relates the partial derivatives of a function f to one another, or to f itself.
- Examples of partial differential equations
  1. $f = 3\,\dfrac{\partial f}{\partial y}$.
  2. The wave equation, $\dfrac{\partial^2 f}{\partial t^2} - c^2 \dfrac{\partial^2 f}{\partial x^2} = 0, \qquad c \in \mathbb{R}$.
  3. The heat equation, $\dfrac{\partial f}{\partial t} = D\dfrac{\partial^2 f}{\partial x^2}, \qquad D > 0$.
- A solution to a partial differential equation is a function f that satisfies the partial differential equation.

Check your understanding

1. Show that the functions $f(x,t) = g(x - c\,t)$ and $f(x,t) = h(x + c\,t)$, where g and h are twice differentiable functions of one variable, solve the wave equation. Such solutions are called traveling wave solutions.

2. Show that the function $f(x,y) = g(x)\exp(y/3)$, where g is an arbitrary function of x, solves the PDE

$$f = 3\,\frac{\partial f}{\partial y}.$$

Check your understanding (continued)

In this problem, we look for a solution to the heat equation

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2}, \qquad D > 0.$$

Consider the function

$$f(x,t) = \frac{1}{2\sqrt{\pi D t}}\int_{-\infty}^{\infty} \exp\left[-\frac{(x-x_0)^2}{4Dt}\right] H(x_0)\, dx_0,$$

where H is a smooth function of x_0.

1. Calculate $\partial f/\partial t$.
2. Calculate $\partial^2 f/\partial x^2$.
3. Show that f formally solves the heat equation given above.