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SHOULD COMPUTER TECHNOLOGY CHANGE THE WAY WE TEACH?

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Introduction

An article in UME Trends, [5], noted four ways students learn mathematics: spontaneously, inductively, constructively, pragmatically. In a typical mathematics lecture, one presents the motivation, theorem, proof, and example as a "fait accompli", with only moderate, if any, student interaction. Those who use this method probably think they are helping the students learn inductively, but most likely this approach helps only spontaneous learning. Computer technology is a valuable aid in helping an instructor of mathematics address the pragmatic and constructive modes of learning, enabling the instructor to be a facilitator of a student's thought process, and not simply a source of knowledge. It allows the classroom emphasis to shift from ensuring that students can replicate what they have been shown to concentrating on helping them with their own "mental schemas". That is, to help them fit new ideas into their previously accepted mathematical truths. Whether we like it or not, the long term effect of our teaching is what students understand about the subject matter, not what we had them memorize.

Some examples will be given here to illustrate how computer technology can change the way we teach. They are applicable to a beginning single variable calculus course and describe a situation where each of the students has immediate access to a computer during the "lecture".

Derivatives

One of the many challenges faced by a calculus instructor is how to prevent students from thinking of a derivative of a function as simply an entry in a list of formulas. A computer or graphing calculator is a very useful aid in presenting the idea of a derivative in several different ways. One way is to plot, on the same screen, the graph of a function $f(x)$ and the graphs of an associated function $[f(x+h)-f(x)]/h$ for a sequence of values of h (say .1, .01, .001, .0001). It becomes clear to the student that as h becomes small, the graph of $[f(x+h) - f(x)]/h$ does not change, i.e. the value of h (h being sufficiently small) is immaterial. Thus the limit is the end product of a process, not the result of plugging in a number (zero), and results in a new function (usually). Figure 1 shows this process for $f(x) = \sin x$, $h = 1, .5, .01, .001$. Having both $f(x)$ and $f'(x)$ on the screen allows the instructor to emphasize the equality of "the slope of the tangent line to f at a point" and "the value of f' at that point." (Note: by setting $a = 0$ and $c = 1$

on the computer screen shown in Figure 1 gives graphical verification that the derivative of the *sin x* is *cos x*.)

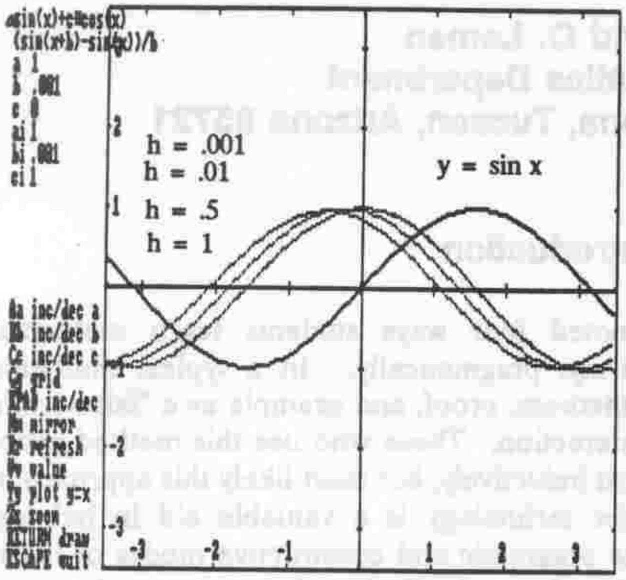

Figure 1: *sin x* and *(sin(x + b) - sin(x))/b*

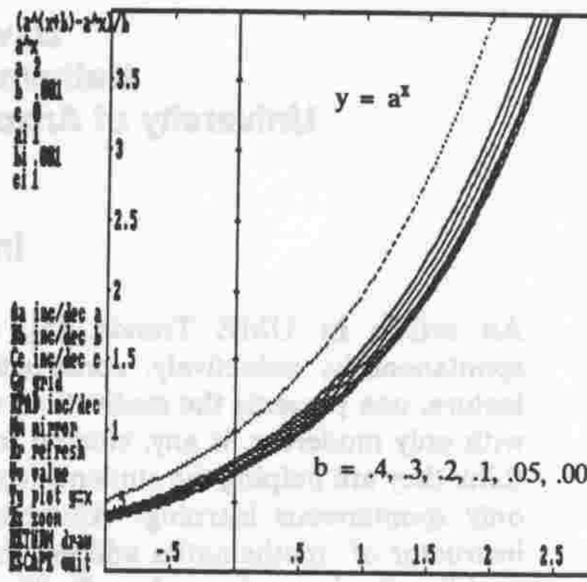

Figure 2: a^x and $[a^{x+b} - a^x]/b$

Performing this process on $f(x) = a^x$, see Figure 2, allows the student to note that the limit process gives a result that is proportional to the original function (you show this analytically). Then the question arises, almost naturally: Can you find a value of *a* such that this proportionality constant is one?". In a situation where each student can experiment with these graphs, the prior activities lead to a lively classroom discussion, with surprising thought provoking comments.

An alternative approach to the derivative is to draw the graph of *f(x)* along with the secant lines through *(b,f(b))* given by

$$y = f(b) + [f(b+h)-f(b)/h][x-b].$$

Letting *h* approach zero, for a specific value of *b*, allows the student to visualize tangent lines as limits of secant lines. This may be done for as many values of *b* as desired.

For an example of computer assistance with the derivative as instantaneous rate of change and as a function, see [1].

Limits and Continuity

Now consider the function $x^n \sin(1/x)$ for $n = 0, 1, 2, 3$ as *x* approaches zero. Obvious questions to consider concern the existence of a limit, and when you have such a limit and define the function to be zero at $x = 0$, the existence of its derivative at $x = 0$.

Also, you can explore continuity questions about the function and its derivative at $x = 0$. The analysis in this case is not made easier by looking at pictures, but the reasons for the appropriate value of n can be made apparent with the following experiments on the computer.

The non-existence of the limit of $sin(1/x)$ as x approaches zero is clearly demonstrated by zooming in on its graph near the origin. The fact that this limit is zero for $x\ sin(1/x)$ is easily demonstrated in the same manner. Now plot on the same screen the graph of the function $x\ sin(1/x)$ and that of some secant lines from the origin. This is done by writing the equation of the secant line from the origin to an arbitrary point on the graph of $x\ sin(1/x)$ and letting this arbitrary point approach the origin in a sequence of discrete steps. Having a zoom feature is a big help in demonstrating that $x\ sin(1/x)$ does not have a derivative at the origin. (See Figure 3, noting that *3a* has the domain $0 \le x \le 1$ while *3b* has the domain $0 \le x \le .1$. The secant lines in *3a* (top to bottom in order) are for $x = .13, .45, .4, .003, .3, .02, .01, .005, .2$. The secant lines in *3b* are for *.049, .0000008, .003, .02, .01, .005, .00002* respectively. Repeating this for the graph of

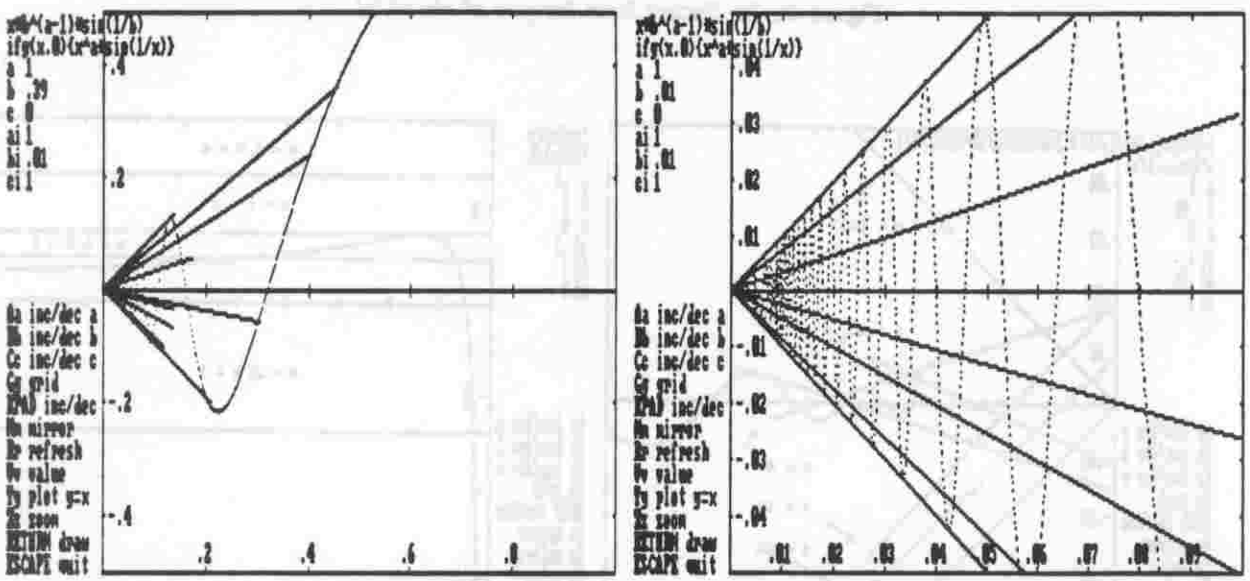

Figure 3a, b: Secant lines for $y = x \sin(1/x)$

x^2 sin(1/x) and a sequence of secant lines from the origin shows the existence of a derivative at the origin (see Figure 4). The continuity of this derivative is investigated by drawing the graph of $x^2\ sin(1/x)$ and a sequence of tangent lines to this graph as the point of tangency approaches the origin. Results of this process are shown in Figure 5. The labels $x =$ on the figure are where the tangent line is drawn. Repeating this process for $x^3\ sin(1/x)$ demonstrates that this function has a continuous derivative at the origin. Some of our instructors used this as the basis for a group discussion during a class period, others had the students come to a computer laboratory and work through a project on these ideas. We also developed an animated "computer demonstration" with these functions, $n = 1, 2, 3$, watching the secant lines and tangent lines as you approached the origin.

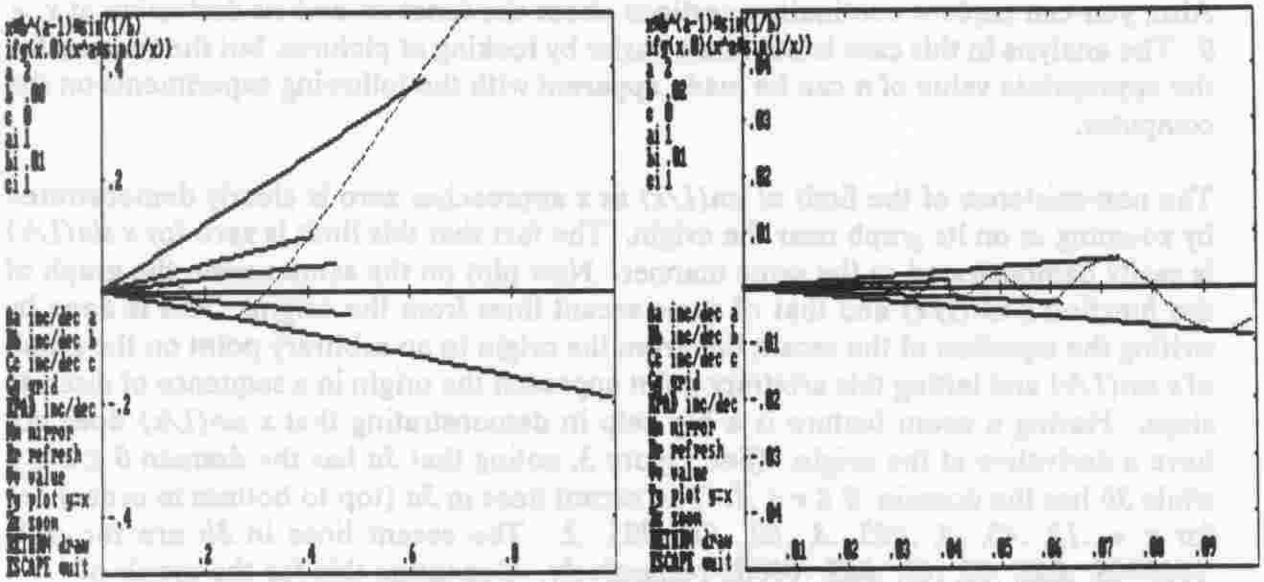

Figure 4a, b: Secant lines for $y = x^2 \sin(1/x)$

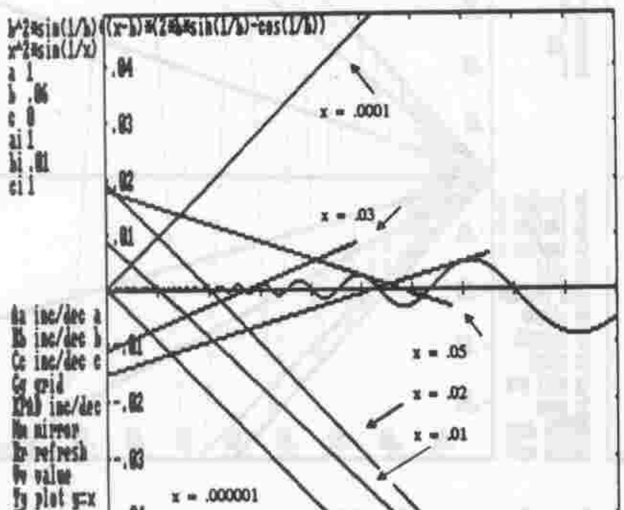

Figure 5: Tangent lines to $y = x^2 \sin(1/x)$

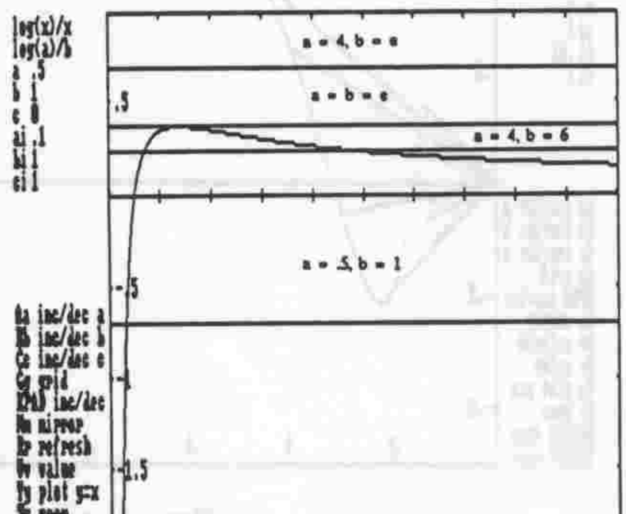

Figure 6: Graphs of *log(x)/x* and *log(a)/b*

Functions

Simply graphing the functions a^x and x^b for several values of a and b is not of great use in determining how the number of curve crossings depend upon a and b. However taking the logarithm of

$$a^x = x^b$$

gives you the equation

$$log(x)/x = log(a)/b.$$

The graph of both sides of this equation is shown in Figure 6, with obvious conclusions. Note that the maximum value of $log(x)/x$ as well as the domains where this function is increasing and where it is decreasing are needed to be certain of the number of crossings of the original curves. Again, doing this as a class project resulted in a very lively discussion.

Games

One of the toolkits that works very well in promoting in class discussion is FINDPOLY, developed by David Lovelock (see [3]). One of its features has the user discover the integer coefficients of an unknown polynomial (of degree less than or equal to seven). The user can ask to see the graph of the polynomial, or that of its first or second derivative. The user can also ask for the value of the polynomial for specific points in its domain or the value of a zero. (The user must bracket the zero in this option.) An interesting aspect of this activity is that there is no "best" strategy for finding these coefficients. Students become very creative during this activity, and it is often the contemplative person that comes up with the best suggestion.

Experiments

The use of IBM's "Personal Science Laboratory" in the classroom also leads to more interactive classroom "lectures". For example a temperature probe was inserted into ice water to determine the decay constant in the equation

$$y = b\ exp(-c\ x) + a$$

which describes Newton's law of cooling. Our software permits the loading of the temperature data directly into a package (TWIDDLE), which permits easy editing as well as fitting a function containing three parameters to the data. The result of one such experiment is shown in Figure 7. The vertical scale reads degrees Celsius and the horizonal scale is in seconds. The data and curve fit for an experiment to determine the terminal velocity of a falling cardboard box are shown in Figure 8. Experiments regarding falling bodies and relating graphs of distance vs. time to horizonal motion engage the students in interesting activities that result in considerable mathematical discussion. Such activities also increase the opportunity for critical thinking in meshing experimental data to mathematical models.

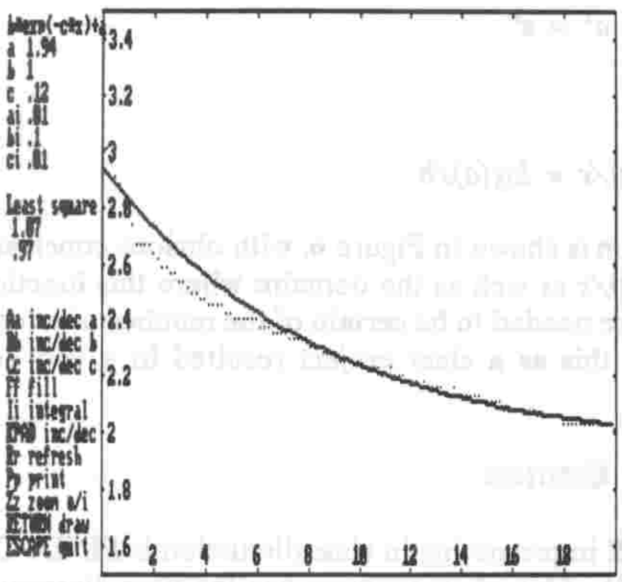

Figure 7: Newton's law of cooling

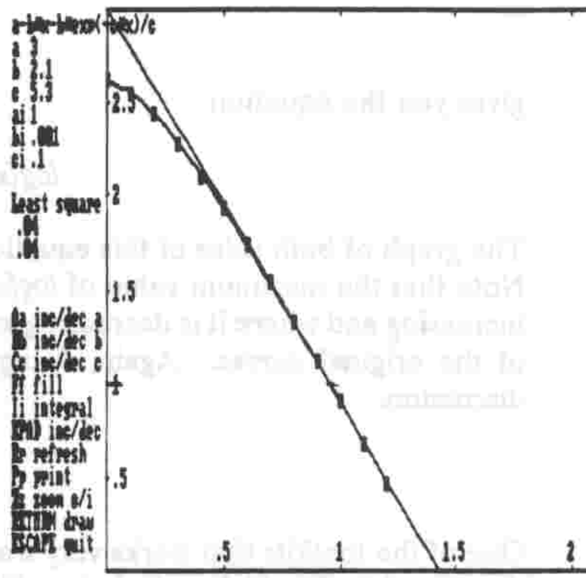

Figure 8: Terminal velocity

Projects

We have developed a series of projects, [4], to aid in our teaching efforts. The one listed in the Appendix 1 is often used as a class project at the end of the first semester of calculus to bring together several main ideas.

Summary

Rather than asking the question posed in the title of this paper, we should ask "How can we take full advantage of computer technology to change the way we teach?". After all, it is a great aid in:

- Emphasizing local and global behavior of functions
- Explaining the interplay between continuous and discrete reasoning
- Encouraging the formulating of conjectures
- Providing a means to check an answer (allows alternative approaches)
- Helping implement problem solving strategies
- Promoting critical thinking.

Computer technology is a powerful aid in helping students think clearly and logically about mathematics, and see how mathematics is the key to a quantitative description of science and the world. Note that all of the above facilitate the goal of having students be actively involved in their own learning. How can we resist this! More ideas for computer use in mathematics instruction are given in [1], [2] and [3].

Acknowledgements and Concluding Remarks

One reason many of the topics in this paper arose was because we are using calculus materials being written by mathematicians at eight different schools, University of Arizona, Chelmsford High School, Colgate, Harvard, Haverford, Southern Mississippi, Stanford, and Suffolk Community College, the so called "Harvard Calculus Consortium". This material emphasizes understanding and de-emphasizes memorization. The development is quite pragmatic and is written for students to read. Thus much of the class time may be used for discussion. The instructors (David Lovelock, Warren May, Bill McCallum, Wayne Raskind, Fred Stevenson, and myself) at the University of Arizona who piloted this material meet once a week to discuss progress, successes and failures. Some of the ideas presented in this paper are a result of these discussions, plus discussions with colleagues, Jim Clay and Bill Velez, who used these materials the following semester.

The computer classroom mentioned in this paper is outfitted with thirty MS-DOS personal computers, one for each student. The software used by the students in this classroom was developed at the University of Arizona, [3], as an aid for students to use computers without "mastering a new language". It takes an average student about five minutes to learn how to use the first toolkit. Since all of the twenty-forty toolkits we have developed operate in the same way, there is no additional time needed to learn how to use subsequent toolkits. This software is in the public domain and may be obtained by anonymous file transfer protocol.

At the end of the two semester calculus course, taught in this computer classroom using the Harvard Calculus Consortium Materials, each student was asked to write a "letter to their cousin" who was to enter the University of Arizona as a freshman. This letter was to contain advice on how to succeed at a calculus course taught in this new manner. Excerpts from these letters are given in Appendix 2. While it is difficult to sort out separate effects of computer technology, the Harvard Calculus Consortium material, and a different teaching style, it is clear that their comments are not what you would usually expect from students on how to succeed in calculus.

References

[1] Cushing, Jim, David Gay, Larry Grove, David Lomen, and David Lovelock (1992), THE ARIZONA EXPERIENCE: Software Development and Use, in F. Demana, B. K. Waits, J. Harvey (Eds.), Proceedings of the Third Annual Conference on Technology in Collegiate Mathematics. Addison Wesley. pp. 41-47.

[2] Lomen, David and David Lovelock (1991), Computer Enhanced Mathematics Education, in F. Demana, B. K. Waits, J. Harvey (Eds.) Proceedings of the Second Annual Conference on Technology in Collegiate Mathematics. Addison Wesley. pp. 237-240.

[3] Lomen, David and David Lovelock (1991), A Manual for the Mathematical Toolkits Software developed by the University of Arizona, University of Arizona, Tucson (16 pages).

[4] Lomen, David, David Lovelock, and William McCallum (1991) Calculus Projects, Department of Mathematics, University of Arizona, Tucson (46 pages).

[5] Seldon, Annie and John Seldon (1990) Constructivism in Mathematics Education: A View of how People Learn, UME Trends, March 1990, American Mathematical Society, Providence (p. 8)

Appendix I: The Road and the River

INFORMATION

A particular river follows the quadratic x^2, while a nearby road follows the cubic

$$ax^3 + bx^2 + cx + d.$$

There are three bridges on the road and they cross the river at (-1, 1), (0, 0), and (1, 1). (The units are miles.) The bridge at (1, 1) is at right angles to the river. A dam is located at (1, 2). The dam is not on this river. There is a spillway from the dam to the nearest point on the river. The spillway is straight.

PROJECTS

Project 1. What are the values of a, b, c, and d?

Project 2. What are the distances between the bridges, as the crow flies?

Project 3. At what angles (in radians and degrees) do the other two bridges cross the river?

Project 4. Where does the spillway from the dam intersect the river?

Project 5. A straight road is to be built from the existing road to the dam. Where, on the original road, should it start to have minimum length?

Project 6. A straight road is to be built from the existing road to the dam. Where, on the original road, should it start to have minimum length, and not cross any water?

Project 7. A farmer owns the land between the river and the road, bounded by the first and last bridges. How many square miles does the farmer own?

Appendix II:
Quotes From Student Essays About First Semester Calculus

"This is not high school plug-and-chug, nobody cares if you can follow patterns. ... You'll study less in this class but think much more."

"Numerical values for answers are nowhere near as important as the method you use to get to your answer and understanding the concept."

"The thing that is different is that you are required to think, not just perform meaningless mathematical operations, where you don't even know why you are doing them."

"Learning the concepts, applying the concepts, developing critical thinking skills and problem solving methods should be your top priorities."

"Your success depends on having a strong work ethic, discipline, critical thinking, listening and communication skills."

"Work through the material, knowing that the (resulting)power bestowed in you will be ten times more than the required effort, which at the moment seems tremendous."

"Since the thought process is so important, one should try to be involved in the lecture as much as possible."

"The task is to be flexible in problem solving and to make the connection between common problems and the symbolic language of math."

"Sit near the front, it is a medically proven fact that having a teacher within a five foot radius of a student provides a temporary cure for narcolepsy." "Pay attention to what the teacher is saying and writing rather than trying to copy down everything from the board." "Take part in the discussions, the concepts can be more fully understood this way." "Jethro, brace yourself; they are going to ask you to think in this class!"