Magnetic oscillations in graphene

Simon Becker (joint work with Maciej Zworski)

Cambridge

Hexagonal quantum graph

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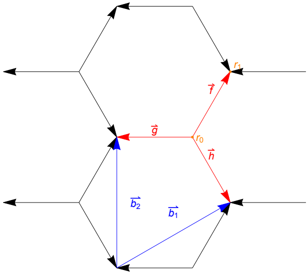

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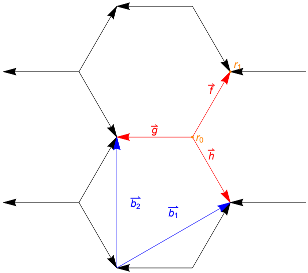

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Kuchment–Post '07

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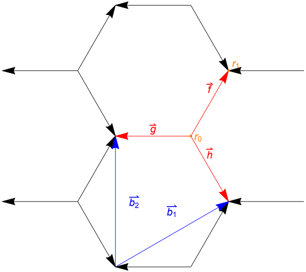

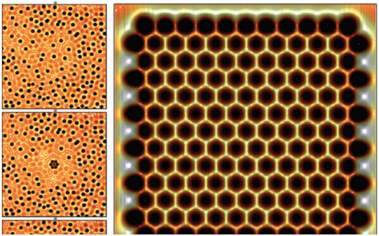

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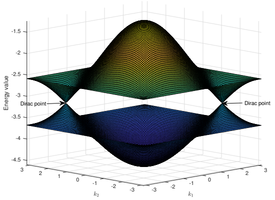
Kuchment–Post '07 Manoharan et al '12



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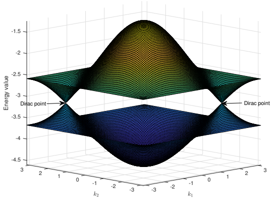

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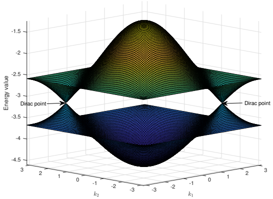

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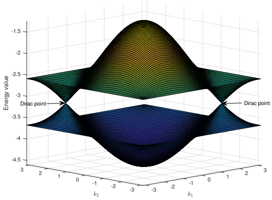

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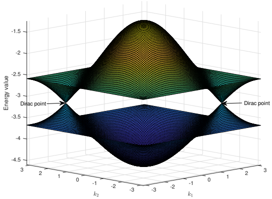

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Fefferman–Weinstein '12, '14: 2D Schrödinger equation models

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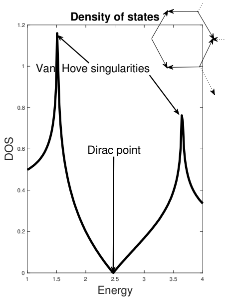

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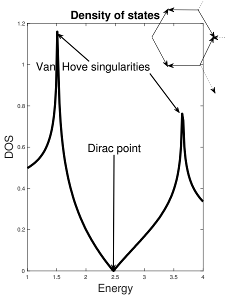

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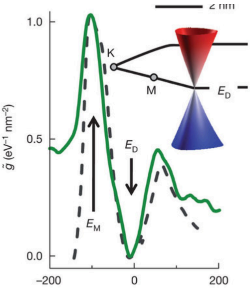

Molecular graphene Manoharan et al '12

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Brüning–Geyler–Pankrashkin '07

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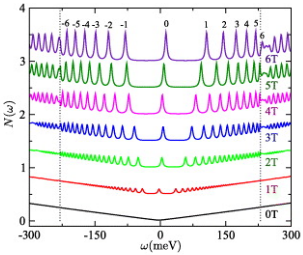

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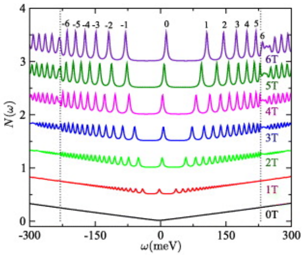
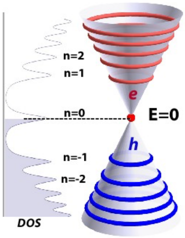


Pound et al '11,



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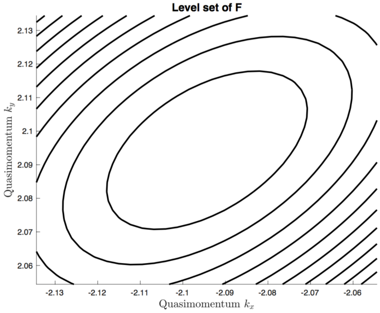

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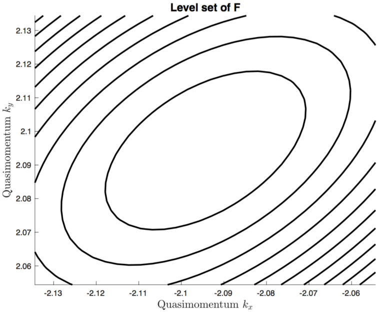

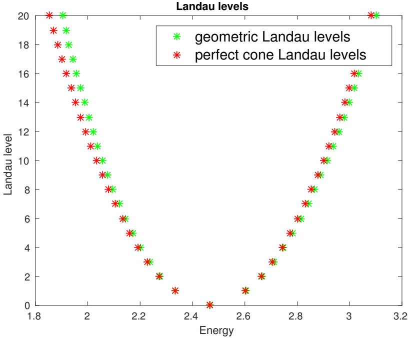

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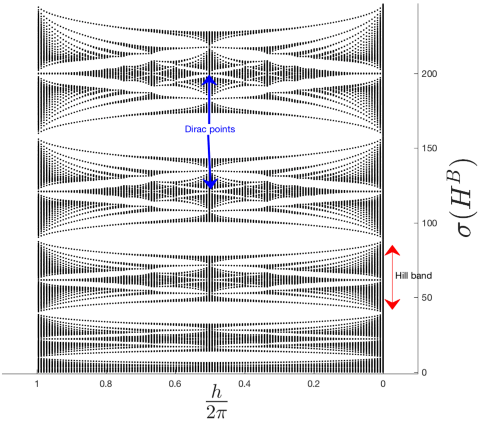

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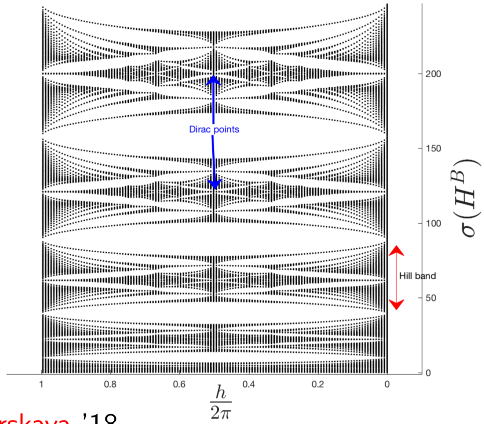

B.–Han–Jitomirskaya '18

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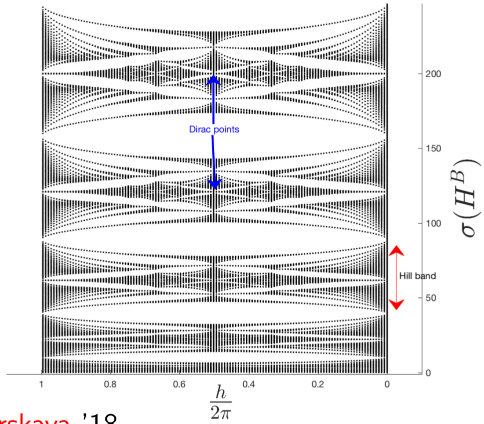

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Hofstadter '76 ... Avila–Jitomirskaya '09 ...

Magnetic (de Haas–van Alphen?) oscillations

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Differentiation can be justified for $\beta < h^{-M}$ (Helffer–Sjöstrand '90)

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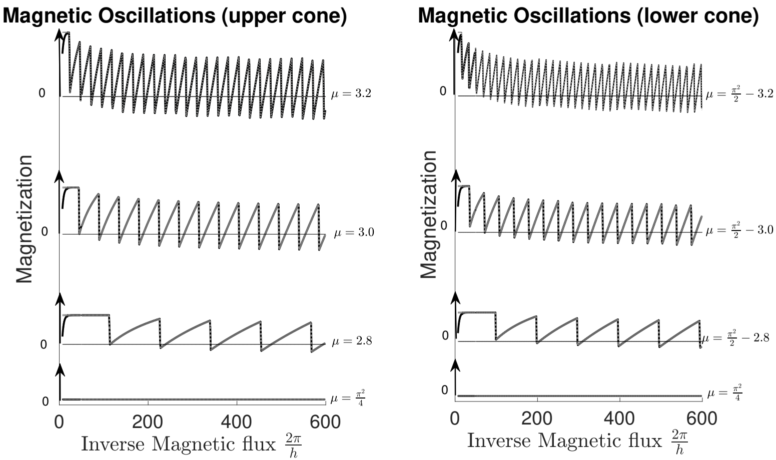

Thank you very much!

S.B. and Maciej Zworksi, (2018), Magnetic oscillations in a model of graphene, arXiv:1801.01931.

S.B., Rui Han, and Svetlana Jitomirskaya, (2018), Cantor spectrum of graphene in magnetic fields, arXiv:1803.00988.