Disordered Quantum Spin Chains
or
"Scratching at the Surface of Many Body Localization"

Günter Stolz
University of Alabama at Birmingham

Arizona School of Analysis and Mathematical Physics
Tucson, Arizona, March 5–9, 2018

Contents

I. Prelude: Towards Many-Body Localization

Localization $\approx$ **Absence of Quantum Transport**

One Body Transport:

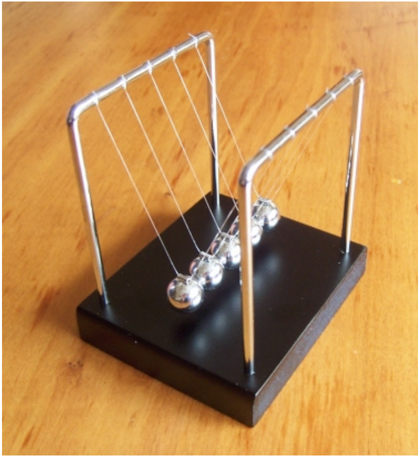
https://www.youtube.com/watch?v=6BLtqLL1fTA

Many Body Transport:

https://www.youtube.com/watch?v=7qPvmYbfM6I

One-Body Transport:

Many-Body Transport:



Want to get a paper into a really good journal?

Take the Heisenberg model (for example):

$$H = \sum_{j \in \mathbb{Z}} (\sigma_j^X \sigma_{j+1}^X + \sigma_j^Y \sigma_{j+1}^Y + \sigma_j^Z \sigma_{j+1}^Z)$$

Add a random field, e.g.

$$H(\omega) = H + \sum_{j \in \mathbb{Z}} \omega_j \sigma_j^Z$$

(i) Show that $H(\omega)$ is "many-body localized" in an "extensive energy regime".

(ii) Probably easier: Show that $H(\lambda\omega)$ is "fully many-body localized" at large disorder λ.

Some words of caution:

- In dimension $d > 1$ **not even the one-particle case is settled**: There is no rigorous proof of extended states for Anderson model in $d = 3$.
- There is no rigorous proof of non-existence of a mobility edge for Anderson model in $d = 2$.
- Hope: Starting with spin chains ($d = 1$) will help. Also, spins are much simpler than electrons.
- However:
  - Dynamical systems methods (as for 1D ergodic Schrödinger operators) will not be equally useful for 1D many body models.
  - In fact: For disordered Heisenberg model a many-body localization/delocalization transition is expected in 1D (for small disorder).

So what **can** we do?

- Will present some first results towards MBL in relatively simple models and regimes ("scratching the surface").
- Focus on getting a sense of phenomena and relevant concepts.
- Have nothing to say about "thermalization" at high energy and the many-body Anderson transition.
- Have nothing to say about multidimensional disordered quantum spin systems.
- Conclusion will be: Field of localization/delocalization for many-body models is still wide open![1]

[1] But be ready to be ridiculed by physicists...

A first (rough) glance at possible manifestations of MBL:

- **Dynamical MBL:** No many-body/information transport (e.g. Newton's cradle), quantum version can be formulated as **zero velocity Lieb-Robinson bound**.
  Spin systems are ideal models to study this (no one-particle transport).

A first (rough) glance at possible manifestations of MBL:

- **Dynamical MBL:** No many-body/information transport (e.g. Newton's cradle), quantum version can be formulated as **zero velocity Lieb-Robinson bound**.
  Spin systems are ideal models to study this (no one-particle transport).
- **Localization of eigenstates (thermal states, etc.):** All eigenstates of a many-body localized system should be "close" to the eigenstates of a non-interacting many-body system, i.e. product states.
  Can be detected through:
  - Rapid decay of correlations
  - Low entanglement (area laws)

II. MBL Properties of the Disordered XY Chain

(In particular: Introduction of Relevant Concepts)

Survey of some of the results of:

- Hamza/Sims/St. 2012
- Klein/Perez 1992
- Sims/Warzel 2016
- Pastur-Slavin 2014
- Abdul-Rahman/St. 2015
- Abdul-Rahman/Nachtergaele/Sims/St. 2016, Survey 2017

XY (or XX) Chain in Random Field:

$$H_{XY}(\omega) = -\sum_{j=1}^{L-1}(\sigma_j^X\sigma_{j+1}^X + \sigma_j^Y\sigma_{j+1}^Y) - \sum_{j=1}^{L}\omega_j\sigma_j^Z \quad \text{in } \bigotimes_{j=1}^L \mathbb{C}^2$$

Assume: $(\omega_j)_{j=1}^{\infty}$ i.i.d. random variables, with distribution $d\mu(\omega_j) = \rho(\omega_j)\,d\omega_j$, where ρ is bounded and compactly supported.

Jordan-Wigner transform:

$$c_1 := a_1, \quad c_j := \sigma_1^Z \dots \sigma_{j-1}^Z a_j, \quad j = 2, \dots, n, \quad a := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Canonical anti-commutation relations (CAR):

$$\{c_j, c_k^*\} = \delta_{jk} I, \quad \{c_j, c_k\} = \{c_j^*, c_k^*\} = 0$$

Then

$$H_{XY} = 2c^* Mc + E_0$$

with $E_0 = -\sum_j \omega_j$, $c = (c_1, \dots, c_L)^t$, $c^* = (c_1^*, \dots, c_L^*)$ and *effective Hamiltonian*

$$M = \begin{pmatrix} \omega_1 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & \omega_L \end{pmatrix}$$

1D Anderson Model!

Remarks: (i) In language of second quantization:

$$H_{XY} \cong 2d\Gamma_a(M) + E_0 \quad \text{on } \mathcal{F}_a(\ell^2([1, L]))$$

(ii) Jordan-Wigner transform is non-local.
(iii) Spin chain has commuting degrees of freedom, while $\{c_j\}$ are anti-commuting degrees of freedom.

Nevertheless we will see:

Anderson localization for $M \Longrightarrow$ MBL for H_{XY}

Known strong form of Anderson Localization:

Localization if Eigenvector Correlators:

$$\mathbb{E}\left(\sup_{|g|\leq 1} |(g(M))_{jk}|\right) = \mathbb{E}\left(\sum_{\ell} |\varphi_\ell(j)||\varphi_\ell(k)|\right) \leq Ce^{-\mu|j-k|}$$

uniformly in L (φ_ℓ eigenvectors of M).

In particular: **Dynamical (One-Body) Localization:**

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} |(e^{-itM})_{jk}|\right) \leq Ce^{-\mu|j-k|}$$

This implies various MBL properties for the disordered XY chain:

Zero-velocity Lieb-Robinson bound (Dynamical MBL)

Local observables: For $A, B \in \mathbb{C}^{2\times 2}$ let
$A_j = I \otimes \ldots \otimes A \otimes \ldots \otimes I$ (in j-th position), $B_k = \ldots$
Heisenberg dynamics: $\tau_t(A_j) = e^{itH} A_j e^{-itH}$

Theorem 1 (Hamza/Sims/St. 2012)
There exist $C < \infty$ and $\mu > 0$ such that

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} \|[\tau_t(A_j), B_k]\|\right) \le C\|A\|\|B\|e^{-\mu|j-k|}$$

for all L, $1 \le j, k \le L$, $A, B \in \mathbb{C}^{2\times 2}$.

Note: Requires averaging over disorder $\mathbb{E}(\cdot)$.

Compare: Lieb-Robinson 1972 (recent explosion of improvements, extensions and applications: Nachtergaele/Sims 2006, Hastings/Koma 2006, Hastings 2007,...):

For a quite general class of quantum spin systems (e.g. with bounded coefficients and bounded interaction range) it holds that

$$\|[\tau_t(A_j), B_k]\| \leq C\|A\|\|B\|e^{-\mu(|j-k|-v|t|)}$$

$v < \infty$ Lieb-Robinson (group) velocity

Deterministic result!

Note: There are results on **anomalous LR-bounds** (quasi-periodic field: Damanik/Lemm/Lukic/Yessen 2014)

Key to **proof of Theorem 1**: Use basic properties of quasi-free and free Fermion systems to show

$$\tau_t(c_j) = \sum_\ell \left(e^{-2iMt}\right)_{j\ell} c_\ell$$

Thus: One-particle dynamics e^{-2iMt} determines many-body dynamics $\tau_t(c_j)$.

Now "undo" Jordan-Wigner by two geometric summations to get dynamics of local operators such as a_j, a_j^*.

Build other local observables from this.

Exponential Decay of Correlations ("Exponential Clustering"):

Theorem 2 (Sims/Warzel 2016)
There exist $C < \infty$ and $\mu > 0$ such that

$$\mathbb{E}\left(\sup_{\psi,t} |\langle\psi, \tau_t(A_j)B_k\psi\rangle - \langle\psi, A_j\psi\rangle\langle\psi, B_k\psi\rangle|\right) \leq C\|A\|\|B\|e^{-\mu|j-k|}$$

for all L, $1 \leq j,k \leq k$, and all $A, B \in \mathbb{C}^{2\times 2}$.
Here the supremum is taken over **all** normalized eigenfunctions ψ of H and all $t \in \mathbb{R}$.

Notes: (i) Same for thermal states $\rho_\beta = e^{-\beta H}/\mathrm{Tr}e^{-\beta H}$, with expectations defined as $\langle A\rangle_{\rho_\beta} = \mathrm{Tr}\rho_\beta A$.
(ii) Earlier related work (ground state): Klein/Perez 1992

Area Law for the Entanglement Entropy:

Bipartite decomposition:

$$\mathcal{H}_L = \mathcal{H}_A \otimes \mathcal{H}_B, \quad \mathcal{H}_A = \bigotimes_{j=1}^{\ell} \mathbb{C}_j^2, \quad \mathcal{H}_B = \bigotimes_{j=\ell+1}^{L} \mathbb{C}_j^2$$

ψ normalized eigenstate of H, $\rho_\psi = |\psi\rangle\langle\psi|$, reduced state:

$$\rho_\psi^A = \mathrm{Tr}_B\, \rho_\psi$$

Bipartite entanglement entropy:

$$\mathcal{E}(\rho_\psi) := \mathcal{S}(\rho_\psi^A) := -\mathrm{Tr}\, \rho_\psi^A \log \rho_\psi^A \quad \Big(= \mathcal{S}(\rho_\psi^B) \Big)$$

For generic pure states: $\mathcal{E}(\rho) \sim \ell$ (volume law)

Uniform Area Law:

Theorem 3: (Abdul-Rahman/St. 2015)
There exists $C < \infty$ such that

$$\mathbb{E}\left(\sup_{\psi} \mathcal{E}(\rho_\psi)\right) \leq C$$

for all L and all $1 \leq \ell < L$. Here the supremum is taken over all normalized eigenstates ψ of H.

Notes: (i) Method due to Pastur/Slavin 2014, who proved the area law for the ground state of a disordered d-dimensional quasi-free Fermion system (bound $C\ell^{d-1}$).
(ii) No logarithmic correction in ℓ.
(iii) **Open problem:** Analogue for thermal states w.r.t. log. negativity (Vidal/Werner 2002). Note that $\mathcal{E}(\cdot)$ is NOT a good entanglement measure for mixed states.

Entanglement Dynamics under a Quantum Quench:

Let

- H_A, H_B restrictions of H to A and B
- ψ_A, ψ_B normalized eigenstates of H_A, H_B, $\rho_A = |\psi_A\rangle\langle\psi_A|$, $\rho_B = |\psi_B\rangle\langle\psi_B|$
- $\rho = \rho_A \otimes \rho_B$ (i.e. $\mathcal{E}(\rho) = 0$)
- $\rho_t = e^{-itH}\rho e^{itH}$ (full) Schrödinger dynamics

Theorem 4: (Abdul-Rahman/Nachtergaele/Sims/St. 2016, special case)
There exists $C < \infty$ such that

$$\mathbb{E}\left(\sup_{t,\psi_A,\psi_B} \mathcal{E}(\rho_t)\right) \leq C$$

for all ℓ and L.

Remarks on MBL Hierarchy:

There are general results of the "type":

Zero-velocity LR bound

$\Longrightarrow$ **Exponential clustering**
(Hamza/Sims/St. 2012, Friesdorf et al 2015)

$\Longrightarrow$ **Area Law** (Brandao/Horodecky 2013/2015)

- For disordered XY chains the above "direct" results are stronger.
- Most likely no satisfying converses, I think. But more math left to be understood here.

Main tool in proofs of Theorems 2 to 4:

Quasifree States and Correlation Matrices

Fact: Eigenstates $\rho = \rho_\alpha$, $\alpha \in \{0,1\}^L$, and thermal states $\rho = \rho_\beta$, $0 < \beta < \infty$, of a quasifree Fermion system c^*Mc are quasifree (i.e. expectations of arbitrary products of the c_j and c_j^* can be calculated by Wick's Rule.

Also: The reduced state ρ^A of a quasifree state ρ is again quasifree.

Thus ρ is uniquely determined by its correlation matrix

$$\Gamma_\rho = (\langle c_j c_k^* \rangle_\rho)_{j,k=1}^L$$

and ρ^A by the restricted correlation matrix

$$\Gamma_\rho^A = (\langle c_j c_k^* \rangle_\rho)_{j,k=1}^\ell$$

For **Theorem 2** (Sims/Warzel): Calculate Pfaffians (in clever ways).

For **Theorems 3 and 4** use (Vidal, Latorre, Rico, Kitaev 2003):

$$\mathcal{S}(\rho^A) = -\mathrm{Tr}\,\rho^A \log \rho^A = -\mathrm{tr}\,h(\Gamma_{\rho^A})$$

where $h(x) = x \log x + (1-x)\log(1-x)$. (Reduces dimension from 2^ℓ to ℓ.)

If $\sigma(M) = \{\lambda_j : j = 1, \ldots, L\}$ is simple, then $\Gamma_{\rho_\alpha} = \chi_{\Delta_\alpha}(M)$, where

$$\Delta_\alpha := \{\lambda_j : \alpha_j = 1\}$$

Rest of proof uses that, by Anderson localization of M,

$$\mathbb{E}\left(\sup_\alpha |(\chi_{\Delta_\alpha}(M))_{jk}|\right) \leq Ce^{-\mu|j-k|}$$

Most of the above can be extended to the anisotropic XY chain in random field (at least for large disorder λ):

$$\begin{aligned} H_\gamma &= -\sum_{j=1}^{L-1}((1+\gamma)\sigma_j^X\sigma_{j+1}^X + (1-\gamma)\sigma_j^Y\sigma_{j+1}^Y) - \lambda\sum_{j=1}^{L}\omega_j\sigma_j^Z \\ &= \mathcal{C}^*\tilde{M}\mathcal{C} + E_0 I \end{aligned}$$

Here $\mathcal{C} = (c_1, \dots, c_L, c_1^*, \dots, c_L^*)^t$, $\mathcal{C}^* = (c_1^*, \dots, c_L^*, c_1, \dots, c_L)$,

Block Anderson model: $\tilde{M} = \begin{pmatrix} M & K \\ -K & -M \end{pmatrix}$

$$M = \begin{pmatrix} \omega_1 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & \omega_L \end{pmatrix}, \quad K = \begin{pmatrix} 0 & -\gamma & & \\ \gamma & \ddots & \ddots & \\ & \ddots & \ddots & -\gamma \\ & & \gamma & 0 \end{pmatrix}$$

III. Interlude: The Ising Model

Have seen: Heisenberg XYZ (XXX) model gets MUCH simpler if we drop the Z term!

What happens for the other extreme, i.e., if we drop the XY terms?

Model gets TRIVIAL! Not even quantum :(

Ising model with disorder:

$$H_{Ising}(\omega) = \frac{1}{4} \sum_{j \in \mathbb{Z}} (I - \sigma_j^Z \sigma_{j+1}^Z) + \sum_{j \in \mathbb{Z}} \omega_j \mathcal{N}_j$$

Here: $\mathcal{N}_j = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_j$ **"local number operator"**

Have normalized so that $\Omega = |\ldots \uparrow\uparrow\uparrow \ldots\rangle$ is ground state ("vaccum") with $H\Omega = 0$. Assume here that $\omega_j \geq 0$ for all j.

$H_{Ising}(\omega)$ is diagonal in the product basis

$$\{\varphi_X := \prod_{j \in X} a_j^* \Omega : \ X \subset \mathbb{Z} \text{ finite}\}$$

"Creation operators": $a_j^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_j$ (of down spins/particles)

Note:

$$(I - \sigma_j^Z \sigma_{j+1}^Z)\varphi_X = \begin{cases} 2\varphi_X & \text{if } \{j, j+1\} \in \partial X \text{ (surface of } X) \\ 0 & \text{else} \end{cases}$$

Thus: $H_{Ising}(\omega)\varphi_X = \left(\frac{1}{2}|\partial X| + \sum_{j \in X} \omega_j\right)\varphi_X$

In particular: $\sigma(H_{Ising}(0)) = \{0, 1, 2, \ldots\}$, eigenspace to $E_k = k$:

$$\text{span}\{\varphi_X : X \text{ consists of } k \text{ sep. clusters of down spins}\}$$

$k = 1$: X interval, i..e., φ_X forms a **single down spin "droplet"**

Philosophical Question:

Is $H_{Ising}(0)$ many-body localized?

Pro:

- H has an eigenbasis of product states.
- Dynamics is trivial, in particular $[\tau_t(A), B] = 0$ for all t if $\text{supp}\, A \cap \text{supp}\, B = \emptyset$.

Con:

- Eigenspaces are highly degenerate. There are eigenstates with long range correlations and high entanglement.
- A truly localized regime should be "rather stable" under (small but extensive) perturbations, which is arguably not the case here (as we will see in the example below).

Corresponding 1-particle question: Is the identity operator $I\varphi = \varphi$ Anderson localized in $\ell^2(\mathbb{Z})$? Seems rather silly...

Adding a random field overcomes these questionable issues:

- In finite volume $[-L, L]$ and assuming that ω_j are i.i.d. with absolutely continuous distributions:
  $H_{Ising}^{[-L,L]}(\omega)$ has almost surely non-degenerate spectrum, so that **all** eigenstates are product states.
  In infinite volume spectrum becomes *dense pure point*.
- One way to think of main result of next chapter:
  MBL for $H_{Ising}(\omega)$ is stable under a "natural" extensive perturbation, at least at low energies.

IV. Localization of the Droplet Spectrum in the XXZ Chain

Idea: Perturb around the Ising model by bringing small XX-terms back!

Droplets in the Ising phase of XXZ:

- Starr 2001, Nachtergaele/Starr 2001, Nachtergaele/Spitzer/Starr 2007
- Fischbacher 2013, Fischbacher/St. 2014, 2017

Localization of the Droplet Spectrum:

- Beaud/Warzel 2017a, 2017b
- Elgart/Klein/St. 2017a, 2017b

The disordered infinite XXZ chain

$$H_{XXZ}(\omega) = \sum_{j\in\mathbb{Z}} h_{j,j+1} + \lambda \sum_{j\in\mathbb{Z}} \omega_j \mathcal{N}_j$$

on $\mathcal{H} = \overline{\{\varphi_X : X \subset \mathbb{Z} \text{ finite}\}}$ (Hilbert space completion).

$$\begin{aligned} h_{j,j+1} &= \frac{1}{4}(I - \sigma_j^Z \sigma_{j+1}^Z) - \frac{1}{4\Delta}(\sigma_j^X \sigma_{j+1}^X + \sigma_j^Y \sigma_{j+1}^Y) \\ &= \frac{1}{4\Delta}(I - \vec{\sigma}_j \cdot \vec{\sigma}_{j+1}) + \frac{1}{4}(I - \frac{1}{\Delta})(I - \sigma_j^Z \sigma_{j+1}^Z) \end{aligned}$$

Assume: (i) $\Delta > 1$ "Ising phase" (model frustration free)

(ii) ω_j i.i.d., a.c. distribution, bounded density, support $[0, \omega_{max}]$

(iii) $\lambda > 0$ disorder parameter

Particle number conservation

The subspaces

$$\mathcal{H}_N = \overline{\text{span}\{\varphi_X : |X| = N\}}, \;\; N = 0, 1, 2, \ldots$$

are invariant under $H_{XXZ}(\omega)$. Thus

$$H_{XXZ}(\omega) = \bigoplus_{N=0}^{\infty} H_N(\omega)$$

Identify $\mathcal{H}_N = \ell^2(\mathcal{V}_N)$, where

$$\mathcal{V}_N = \{x = (x_1, \ldots, x_N) \in \mathbb{Z}^N : x_1 < x_2 < \ldots < x_N\},$$

i.e., x are ordered labelings of the down spin configurations $X = \{x_1, \ldots, x_N\} \subset \mathbb{Z}$, $\varphi_X = \delta_x$.

The N-particle operators:

- $H_0(\omega) = 0$ on $\operatorname{span}\Omega$
- $N \geq 1$:

$$H_N(\omega) = -\frac{1}{2\Delta}A_N + \frac{1}{2}D_N + \lambda V_\omega \text{ on } \ell^2(\mathcal{V}_N)$$

Here:

- A_N adjacency operator on $\mathcal{V}_N$ (with ℓ^1-distance inherited from $\mathbb{Z}^N$)
- D_N multiplication operator ("potential") on $\ell^2(\mathcal{V}_N)$ by

$$D_N(x) = 2\,|\{\text{connected components of } X\}| = |\partial X|$$

  (D_N is actually the degree function on the graph $\mathcal{V}_N$, see also talk by C. Fischbacher)
- $V_\omega(x) = \omega_{x_1} + \ldots + \omega_{x_N}$ N-particle Anderson potential

Happy Schrödinger :-)

Free XXZ ($\omega = 0$): $H_{XXZ}(0) = \bigoplus_{N \geq 0} H_N(0)$

System of attractive *hard core bosons* (down spins) at sites $x = \{x_1 < x_2 < \ldots < x_N\}$:

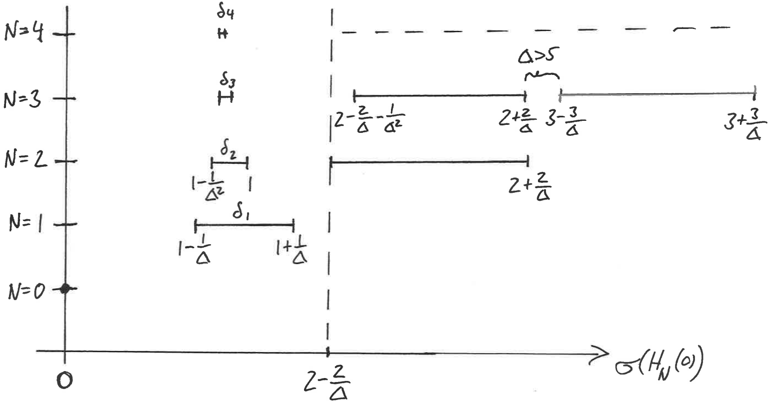
$$H_N(0) = \underbrace{-\frac{1}{2\Delta} A_N}_{\text{hopping (XX)}} + \underbrace{\frac{1}{2} D_N}_{\text{interaction (Ising)}} \quad \text{on } \ell^2(\mathcal{V}_N),\ N \geq 1$$

Attractive interaction: $D_N(x) = N - \sum_{1 \leq k < \ell \leq N} Q(|x_k - x_\ell|)$, where $Q(1) = 1$, $Q(r) = 0$ for $r \neq 1$.

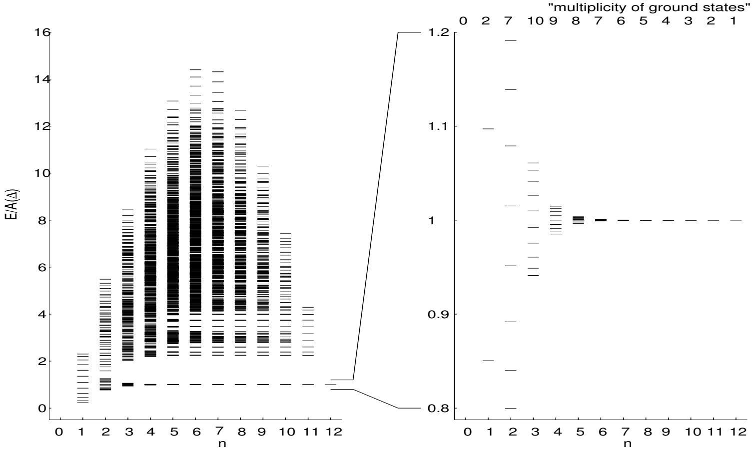
$H_{XXZ}(0)$ (meaning each $H_N(0)$) can be exactly diagonalized by the Bethe Ansatz[2] (Babbitt/Thomas/Gutkin 70's to 90's, Borodin et al 2015):

[2]But our proofs do actually NOT use this!

$\sigma(H_N(0))$ for general N $(\Delta > 3)$



Nachtergaele/Starr 2001, Figure 2 ($\Delta = 2.125$):



Droplet bands: (with $\cosh(\rho) = \Delta$)

$$\begin{aligned} \delta_N &= \left[\tanh(\rho) \cdot \frac{\cosh(N\rho) - 1}{\sinh(N\rho)}, \tanh(\rho) \cdot \frac{\cosh(N\rho) + 1}{\sinh(N\rho)}\right] \\ &\to \quad \delta_\infty = \left\{\sqrt{1 - 1/\Delta^2}\right\} \text{ as } N \to \infty \text{ (exp. fast)} \end{aligned}$$

Droplet spectrum:

$$I = \left[1 - \frac{1}{\Delta}, 2\Big(1 - \frac{1}{\Delta}\Big)\right) =: [E_0, E_1)$$

How do eigenstates to $E \in I$ look like?

They are "exponentially close" to droplets: See exact expressions in Nachtergaele/Starr 2001

The finite volume disordered XXZ chain:

On $\mathcal{H}_{[-L,L]} = \bigoplus_{j=-L}^{L} \mathbb{C}^2$:

$$H_{XXZ}^L(\omega) = \sum_{j=-L}^{L-1} h_{j,j+1} + \lambda \sum_{j=-L}^{L} \omega_j \mathcal{N}_j + \beta(\mathcal{N}_{-L} + \mathcal{N}_L)$$

Here: $\omega_j \geq 0$ for all j, $\beta \geq \frac{1}{2}(1 - \frac{1}{\Delta})$ ("droplet b.c.")

Particle number conservation:

$$H_{XXZ}^L(\omega) = \bigoplus_{N=1}^{2L-1} H_N^L(\omega) \text{ on } \bigoplus_{N=1}^{2L-1} \ell^2(\mathcal{V}_N^L),$$

$$\mathcal{V}_N^L := \{x = (x_1, \ldots, x_N) \in \mathbb{Z}^N : -L \leq x_1 < \ldots < x_N \leq L\}$$

Droplet configurations in $\mathcal{V}_N^L$:

$$\mathcal{V}_{N,1}^L := \left\{ x = (x_1, x_1 + 1, x_1 + N - 1) \in \mathcal{V}_N^L \right\}$$

Theorem 4: (Elgart/Klein/St. 2017a, Fischbacher/St. 2017)

Let $E < E_1$, $H_N^L(\omega)\psi_E = E\psi_E$, $\|\psi_E\| = 1$, and let $\mathcal{A} \subset \mathcal{V}_N^L$. Then

$$\|\chi_{\mathcal{A}}\psi_E\| \leq \frac{4}{E_1 - E} e^{-\frac{E_1 - E}{2}(\log \Delta) d_N(\mathcal{A}, \mathcal{V}_{N,1}^L)} \|\chi_{\mathcal{V}_{N,1}^L}\psi_E\|.$$

Here d_N denotes the ℓ^1-distance of two subsets of $\mathbb{Z}^N$.

- This is a consequence of a *Combes-Thomas bound* on Green's function shown in Elgart/Klein/St. 2017a.
- **Constants are uniform in N.**
  This becomes possible in the Combes-Thomas bound (where the decay rate usually depends on the dimension as $1/N$) because the two terms in $-\frac{1}{2\Delta}A_N + \frac{1}{2}D_N$ "balance" one another, uniformly in N.
- This bound is deterministic. In particular, it holds for $\omega = 0$.

Entanglement bounds: (from now on $H := H^L_{XXZ}(\omega)$)

Bipartite decomposition: $[-L, L] = \Lambda_\ell \cup \Lambda_r$, $\mathcal{E}(\rho_\psi) := \mathcal{S}(\rho_\psi^{\Lambda_\ell})$

Theorem 5: (essentially Beaud/Warzel 2017b)
Let $\Delta > 1$ and $\delta > 0$. Then there exist constants $C_1 = C_1(\Delta, \delta)$ and $C_2 = C_2(\Delta, \delta)$ such that

(a) for every $\omega \geq 0$, every $E \leq E_1 - \delta$ and every normalized eigenvector $\psi_E \in \mathcal{H}_N$ of H to E,

$$\mathcal{E}(\rho_{\psi_E}) \leq C_1 \log \min\{|\Lambda_\ell|, N\} \leq C_1 \log |\Lambda_\ell|$$

(b) If $\mathbb{E}(\cdot)$ denotes disorder averaging, then

$$\mathbb{E}\Big(\sup_{\substack{E \leq E_1 - \delta \\ \|\psi_E\| = 1}} \mathcal{E}(\rho_{\psi_E}) \Big) \leq C_2$$

Remarks:

- Part (a) holds, in particular, for $\omega = 0$ (droplet states of free Ising phase XXZ)! C_1 is quite explicit.
- Averaging over disorder kills the log-correction!
- Part (a) follows from Theorem 4 and summing up many geometric series.
- Part (b) follows quite easily from Part (a), because large down-spin clusters rarely have energy below E_1 (by large deviations for $\sum_{j=1}^{N} \omega_j$).

Need to work harder for LR bounds and correlation bounds!
(Remember MBL hiearchy.)

Key result for everything else to come:

Droplet Localization:

Theorem 6: (Elgart/Klein/St. 2017a)

Let $\delta > 0$, $\lambda > 0$ and $\Delta > 1$ be such that $\lambda\sqrt{\Delta - 1}$ is sufficiently large (dep. on δ). Then there exist $C < \infty$ and $m > 0$ such that

$$\mathbb{E}\left(\sum_{E \in \sigma(H) \cap I_{1,\delta}} \|\mathcal{N}_j \psi_E\| \|\mathcal{N}_k \psi_E\|\right) \leq C e^{-m|j-k|} \qquad (1)$$

uniformly in $L > 0$, $j, k \in [-L, L]$.

Here ψ_E is the (almost surely unique) normalized eigenstate to $E \in \sigma(H)$ and

$$I_{1,\delta} := [E_0, E_1 - \delta]$$

Remarks (including on the proof):

- Interpretation: Eigenstates in the droplet spectrum are close to states with only one down-spin cluster!
- Special cases: (i) $\Delta > 1$ fixed, λ large, (ii) $\lambda > 0$ fixed, Δ large.
- Droplet localization (??) is a form of many-body eigencorrelator localization in the droplet spectrum. In particular:

$$\mathbb{E}\left(\sup_{|g| \leq \chi_{I_{1,\delta}}} \|\mathcal{N}_j g(H) \mathcal{N}_k\|_1 \right) \leq \mathbb{E}\left(\sum_{E \in \sigma(H) \cap I_{1,\delta}} \|\mathcal{N}_j \psi_E\| \|\mathcal{N}_k \psi_E\| \right)$$

Note here that for a simple eigenvalue:

$$\|\mathcal{N}_j |\psi_E\rangle\langle\psi_E| \mathcal{N}_k\|_1 = \|\mathcal{N}_i \psi_E\| \|\mathcal{N}_j \psi_E\|$$

- In particular, this is applicable to the (energy restricted) dynamics of H via $g(H) = \chi_{I_{1,\delta}}(H)e^{-itH}$.
- Particle number conservation reduces the proof to summing over eigencorrelators in the N-particle subspaces.
- In each N-particle space, the bound is proven by the fractional moments method. Uniformity of constants in N follows from uniformity of constants in the Combes-Thomas bound.
- Summability over N follows by a large deviations argument ("IDS" decays exponentially in N).

Consequences of Droplet Localization:

We keep all assumptions on $H = H^L_{XXZ}(\omega)$ from above, including the (Δ, λ)-regime from Theorem 6.

Consequences of droplet localization are **energy-restricted** versions of several of our favorite MBL manifestations.

More precisely:

- $I := I_{1,\delta}$ for any fixed $\delta > 0$ (which constants below will depend on).
- $P_I = \chi_I(H) =$ spectral projection for H onto I

- Heisenberg evolution: $\tau_t(X) = e^{itH} X e^{-itH}$
- Projection of local observable onto energy window: $X_I := P_I X P_I$
- Energy restricted Heisenberg dynamics:

$$\tau_t(X_I) = e^{itH} P_I X P_I e^{-itH} = P_I \tau_t(X) P_I = (\tau_t(X))_I$$

- Compare with energy-restricted Schrödinger dynamics (as routinely used for one-particle models):

$$e^{-itH} P_I \psi, \quad \psi \in \mathcal{H}$$

Zero-velocity Lieb-Robinson bound:

Theorem 7: (Elgart/Klein/St. 2017b)

There exist $C < \infty$ and $m > 0$, independent of L, such that, for all local observables X and Y,

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} \|[\tau_t(X_I), Y_I]\|\right) \leq C\|X\|\|Y\|e^{-m\,\mathrm{dist}(S_X,S_Y)}.$$

Recall $I = I_{1,\delta}$. $X_I = \chi_I(H)\,X\,\chi_I(H)$.

S_X and S_Y are supports of X and Y (assumed to be intervals).

Quasi-locality of the dynamics ("LR for gourmets"):

Theorem 8: (Elgart/Klein/St. 2017b)

There exist $C < \infty$ and $m > 0$ such that for all X, t and ℓ there exists a (random) local observable $X_\ell(t) = X_\ell(t, \omega)$ with

$$S_{X_\ell(t)} = S_{X,\ell} = S_X + [-\ell, \ell]$$

and

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} \|(X_\ell(t) - \tau_t(X))_I\|\right) \leq C\|X\|e^{-m\ell}.$$

Note: In the energy restricted case zero-velocity LR and quasi-locality are NOT equivalent!

Exponential clustering:

Theorem 9: (Elgart/Klein/St. 2017a)

There exist $C < \infty$ and $m > 0$ such that for all local observables X and Y,

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} \sum_{E\in\sigma(H)\cap I} R_{\tau_t(X_I),Y_I}(\psi_E)\right) \leq C\|X\|\|Y\|e^{-m\,\mathrm{dist}(S_X,S_Y)}.$$

Here

$$R_{X,Y}(\psi) = |\langle\psi, XY\psi\rangle - \langle\psi, X\psi\rangle\langle\psi, Y\psi\rangle|$$

Note: According to our proposed "hierarchy of MBL properties" this should be the easiest result to prove.

V. Two illustrative (sketches of) proofs

- The Combes-Thomas bound
- Exponential clustering

A Combes-Thomas bound (in infinite volume)

Fix $N \in \mathbb{N}$ (arbitrary) and drop all N's from notation.

$$\begin{aligned} H = H_N &= -\frac{1}{2\Delta} A + \frac{1}{2} D + V \quad \text{on } \ell^2(\mathcal{V}_N) \\ &= -\frac{1}{2\Delta}\mathcal{L} + \frac{1}{2}(1 - \frac{1}{\Delta}) D + V, \end{aligned}$$

with any potential $V \geq 0$, $\mathcal{L} = A - D$ graph Laplacian on $\mathcal{V}_N$.
Recall:
$D(x) = 2\,|\{\text{connected components of } x\}| = \text{degree of } x \text{ in } \mathcal{V}_N$

$P_1 :=$ orthogonal projection onto $\ell_2(\mathcal{V}_{N,1})$

Thus: $H + (1 - \frac{1}{\Delta})P_1 \geq 2(1 - \frac{1}{\Delta})$.

Theorem 10: (Elgart/Klein/St. 2017a, Fischbacher/St. 2017)

Let $\delta > 0$, $E \leq (2-\delta)(1-\frac{1}{\Delta})$ and $\mathcal{A}, \mathcal{B} \subset \mathcal{V}_N$. Then

$$\|\chi_{\mathcal{A}} \left(H + (1 - \frac{1}{\Delta}) P_1 - E \right)^{-1} \chi_{\mathcal{B}}\| \leq \frac{4\Delta}{\delta(\Delta - 1)} \left(1 + \frac{\delta(\Delta-1)}{8} \right)^{-d_N(\mathcal{A},\mathcal{B})}$$

Remarks: (1) No N-dependence in the constants!!!

(2) Can be extended to complex energy, works in finite volume, etc.

Sketch of proof:

As in the standard proof we use *dilations:*

$$K_\eta := e^{-\eta\rho_{\mathcal{A}}} H e^{\eta\rho_{\mathcal{A}}} - H$$

where $\eta > 0$, $\rho_{\mathcal{A}}(x) := d_N(\mathcal{A}, x)$.

$$(K_\eta\psi)(x) = \frac{1}{2\Delta} \sum_{y \in \mathcal{V}_N : y \sim x} (1 - e^{-\eta(\rho_{\mathcal{A}}(y) - \rho_{\mathcal{A}}(x))})\psi(y)$$

$$x \sim y \Rightarrow \ |1 - e^{-\eta(\rho_{\mathcal{A}}(y) - \rho_{\mathcal{A}}(x))}| \leq e^\eta - 1$$

Standard C-T: $\|K_\eta\| \leq \frac{e^\eta - 1}{2\Delta} \cdot 2N$ (use $2N =$ max. degree of $\mathcal{V}_N$)

Grows with N. Not good enough!

Instead: Borrow two factors $D^{-1/2}$ and show

$$\|D^{-1/2}K_\eta D^{-1/2}\| \le C(\Delta)(e^\eta - 1).$$

N-independence is a consequence of the "balance" of A and D.

The two borrowed factors can be paid back via (due to $\mathcal{L} \ge 0$)

$$\|D^{1/2}(H + \frac{1}{2}(1 - \frac{1}{\Delta})P_1 - E)^{-1}D^{1/2}\| \le C(\delta, \Delta)$$

Now continue as in the standard C-T proof (essentially a resolvent identity, e.g. Kirsch 2007)

Exponential Clustering

Proof of Theorem 9 for $t = 0$ and one-site observables $X \in \mathcal{A}_j$, $Y \in \mathcal{A}_k$, $j \neq k$: Can be built from

$$X^{+,+} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_j, X^{+,-} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_j, X^{-,+} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_j, X^{-,-} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_j$$

$$Y^{\pm,\pm} = \text{ same at site } k$$

16 cases: $R_{X^{a,b},Y^{c,d}}(\psi_E), \quad a, b, c, d \in \{\pm\}$

Case 1: $X^{-,-} = \mathcal{N}_j$, $Y^{-,-} = \mathcal{N}_k$

$$\begin{aligned} R_{\mathcal{N}_j,\mathcal{N}_k}(\psi_E) &= |\langle \psi_E, \mathcal{N}_j P_I (I - |\psi_E\rangle\langle\psi_E|) P_I \mathcal{N}_k \psi_E \rangle| \\ &\leq \|\mathcal{N}_j \psi_E\| \|\mathcal{N}_k \psi_E\| \end{aligned}$$

Thus Case 1 follows from Theorem 6.

Many other cases can be settled by particle number conservation:

$$\text{Assume } \psi = \psi_E \in \mathcal{H}_N \text{ for some } N$$

(holds a. s. for all eigenvectors by simplicity).

$Z \in \{X, Y\} \Longrightarrow$

$$Z^{+,-}\psi \text{ has at most } N-1 \text{ particles } (\in \bigoplus_{j=1}^{N-1} \mathcal{H}_j)$$

$$Z^{-,+}\psi \text{ has at least } N+1 \text{ particles } (\in \bigoplus_{j=N+1}^{2L+1} \mathcal{H}_j)$$

$$Z^{-,-} \text{ and } P_I \text{ preserve the particle number}$$

This settles five more cases:

$$\begin{aligned} 0 &= R_{X^{+,-},Y^{+,-}}(\psi) = R_{X^{-,+},Y^{-,+}}(\psi) = R_{X^{+,-},Y^{-,-}}(\psi) \\ &= R_{X^{-,-},Y^{-,+}}(\psi) = R_{X^{-,-},Y^{+,-}}(\psi) \end{aligned}$$

For example:

$$\underbrace{\langle \psi, X^{+,-} P_I Y^{+,-} \psi \rangle}_{= \langle X^{-,+}\psi, P_I Y^{+,-}\psi\rangle = 0} - \underbrace{\langle \psi, X^{+,-} \psi \rangle}_{=0} \cdot \underbrace{\langle \psi, Y^{+,-} \psi \rangle}_{=0}$$

Another eight cases can be reduced to the previous six cases by using the properties

(a) $R_{X^{+,+},Z} = R_{X^{-,-},Z}$ $(X^{+,+} = I - X^{-,-})$

(b) $R_{Z,Y^{+,+}} = R_{Z,Y^{-,-}}$

(c) $R_{Z,W} = R_{W^*,Z^*}$

This leaves two cases:

$$\text{(i)} \;\; R_{X^{-,+},Y^{+,-}}(\psi), \quad \text{(ii)} \;\; R_{Y^{-,+},X^{+,-}}(\psi)$$

Case (ii) reduces to Case (i) by commutation.

The remaining Case (i):

$$\begin{aligned}\sum_E R_{X^{-,+},Y^{+,-}}(\psi_E) &= \sum_E |\langle \psi_E, X^{-,+} P_I (I - P_E) Y^{+,-} \psi_E \rangle| \\ &= \sum_E |\langle \psi_E, \mathcal{N}_j X^{-,+} P_I (I - P_E) Y^{+,-} \mathcal{N}_k \psi_E \rangle| \\ &\leq \sum_E \|\mathcal{N}_j \psi_E\| \|\mathcal{N}_k \psi_E\|\end{aligned}$$

The claim follows from Theorem 6.

QED

VI. Epilogue: Where to go from here? (Gentle Ben's!)

Goals worthwhile trying:

- Scratch deeper in the Ising phase of XXZ: States with k connected clusters of down-spins?

  Note: Theorem 6 (droplet localization) does NOT hold above $[1 - 1/\Delta, 2(1 - 1/\Delta)]$ (by a Theorem in Elgart/Klein/St. 2017b). But states with k downspin clusters have energy at least $k(1 - 1/\Delta)$.

  Will have to think much more in terms of scattering theory (of k quasi-particles)!

Also worthwhile:

Find some other concrete models where MBL regimes can be identified! (Before attempting results for "classes" of quantum spin systems.)

- Models without particle number conservation?
  Tempting thought: Can the methods for anisotropic XY and XXZ be combined to get a result for fully anisotropic XYZ (In Ising phase)? Unhappy Schrödinger!
- Models with spin $> 1/2$? (Higher spin XXZ, AKLT)
- Models and more models: beyond nearest neighbors, beyond quantum spin systems (harmonic oscillators, Abdul-Rahman 2017), ...

A reasonable attempt at a first result for a "class" of spin systems:

- Show that large disorder leads to a "fully MBL" regime for a general class of spin chains with short range interaction (i.e., complete the program of Imbrie 2016, who uses an unproven assumption on level spacing).

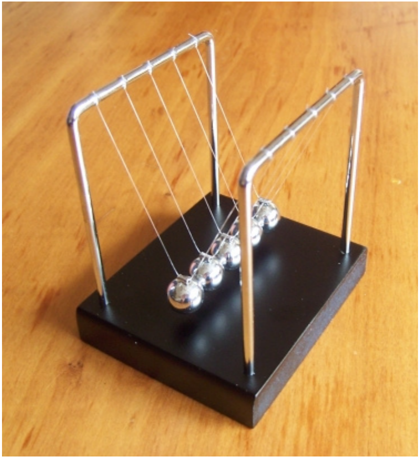
Even here it may initially help to focus on a concrete example such as the XXX Heisenberg chain.

(Imbrie: $\sum_i J_i \sigma_i^Z \sigma_{i+1}^Z + \sum_i h_i \sigma_i^Z + \sum_i \gamma_i \sigma_i^X$)

Homework (unreasonable goals):

- We have shown "zero-temperature localization" for disordered XXZ in the Ising phase.
  MBL in an "extensive energy regime" would mean localization for energies in $[-\rho L, \rho L]$ for some positive particle density $\rho > 0$ (L the system size). May currently be out of reach.
- Thermalization? Many-body mobility edge? Well, maybe better start with proving existence of extended states in 3D Anderson model...
- Higher-dimensional spin systems? Even XY not understood. Will require much more groundwork for the non-random case.
- Electron gases? (Basko/Aleiner/Altshuler 2006)

Cut the wires!



References:

Abdul-Rahman/Nachtergaele/Sims/Stolz 2016, arXiv:1510.00262
Abdul-Rahman/Nachtergaele/Sims/Stolz 2017, arXiv:1610.01939
Abdul-Rahman 2017, arXiv:1707.07063
Abdul-Rahman/Stolz 2015, arXiv:1505.02117
Babbitt/Thomas/Gutkin 70s to 90s
Basko/Aleiner/Altshuler 2006, arXiv:cond-mat/0506617
Beaud/Warzel 2017a, arXiv:1703.02465
Beaud/Warzel 2017b, arXiv:1709.10428
Borodin et al 2015, arXiv:1407.8534
Brandao/Horodecky 2013, arXiv:1309.3789
Brandao/Horodecky 2015, arXiv:1206.2947
Damanik/Lemm/Lukic/Yessen 2014, arXiv:1408.1796
Elgart/Klein/Stolz 2017a, arXiv:1703.07483
Elgart/Klein/Stolz 2017b, arXiv:1708.00474
Fischbacher 2013, Master Thesis, LMU Munich
Fischbacher/Stolz 2014, arXiv:1309.1858
Fischbacher/Stolz 2017, arXiv:1712.10276

Friesdorf et al, arXiv:1409.1252
Hamza/Sims/Stolz 2012, arXiv:1108.3811
Hastings 2007, arXiv:0705.2024
Hastings/Koma, arXiv:math-ph/0507008
Imbrie 2016, arXiv:1403.7837
Kirsch 2007, arXiv:0709.3707
Klein/Perez 1992, CMP 147, 241-252 (1992)
Lieb/Robinson 1972, CMP 28, 251257 (1972)
Nachtergaele/Spitzer/Starr 2007, arXiv:math-ph/0508049
Nachtergaele/Sims 2006, arXiv:math-ph/0506030
Nachtergaele/Starr 2001, arXiv:math-ph/0009002
Pastur/Slavin 2014, arXiv:1408.2570
Sims/Warzel 2016, arXiv:1509.00450
Starr 2001, arXiv:math-ph/0106024
Vidal/Latorre/Rico/Kitaev 2003, arXiv:quant-ph/0211074
Vidal/Werner 2002, arXiv:quant-ph/0102117