Arizona School of Analysis and Mathematical Physics, Tucson, 7 March 2018

Stability of the superselection sectors of Kitaev's abelian quantum double models

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[1]Based on work supported by the U.S. National Science Foundation under grant # DMS-1515850.

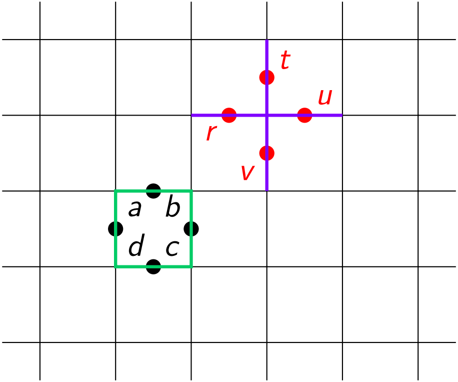
Outline

- Kitaev's quantum double models — toric-code model
- Ground states and excitations
- Infinite systems, GNS representation, (in)equivalent representations
- Stability of superselection sectors

Kitaev's quantum double models (Kitaev, 2003)

To save time we focus in these slides on the Toric Code model (TCM). There is a similar model for any finite group G ($G = \mathbb{Z}_2$ for TCM), and everything generalizes to arbitrary abelian G and many results also hold for non-abelian G.

$\mathcal{H}_x = \mathbb{C}^2$ for all $x \in \mathcal{E}(\mathbb{Z}^2)$, the edges of the square lattice, and we are interested in the infinite-volume model.



$$H = \sum_s (\mathbb{1} - A_s)$$

$$+ \sum_p (\mathbb{1} - B_p)$$

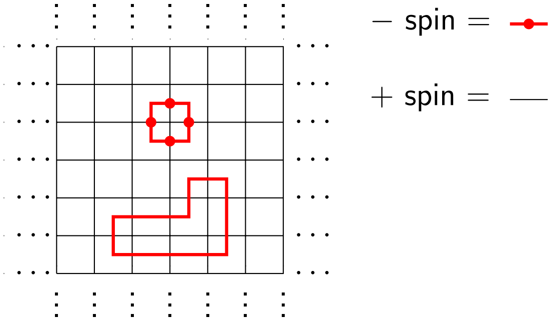
$$A_s = \sigma^1_r \sigma^1_t \sigma^1_u \sigma^1_v$$

$$B_p = \sigma^3_a \sigma^3_b \sigma^3_c \sigma^3_d$$

As shown by Kitaev, the dimension of the ground state space of the TCM on a finite torus (periodic b.c.), is 4, and more generally on a surface of genus g, the dimension is 4^g. The ground state energy is 0, i.e., these are frustration free ground states.

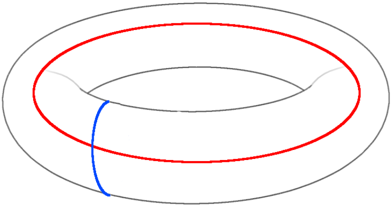
Why?
All A_s and B_p commute. The joint kernel of the operators $\mathbb{1} - B_p$ is spanned by spin configurations that have an even number of $-$ spins on each plaquette, and each A_s flips the spins on four edges meeting in a vertex, which leaves that kernel invariant.

It is useful to represent the spin configurations as a set of paths in the dual lattice:



Note that spin configurations obtained by acting with A_s's on the all $+$ configuration, are described by closed loops.

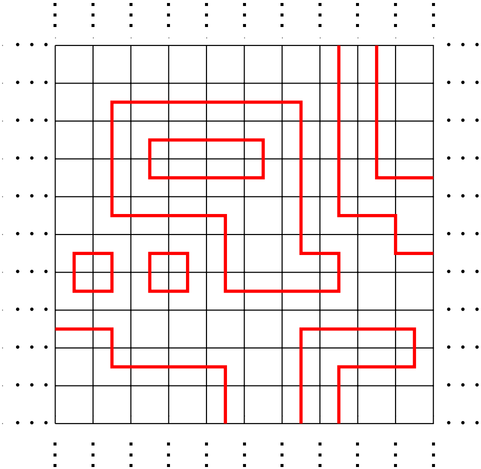
Now, it is clear why we get 4 ground states on the torus:



Single non-trivial loops such as the red or the blue circle cannot be created by the action of the operators A_s.
Therefore there are classes of spin configurations and the equal-coefficient superpositions of all configurations within each class gives a ground state.
For a detailed discussion of the more general class of abelian quantum-double models see Bachmann, 2017.

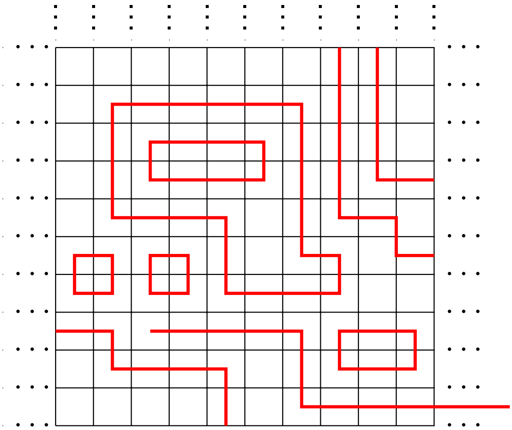
On all of $\mathbb{Z}^2$, the model has a unique frustration free ground state (Alicki-Fannes-Horodecki, 2007): there is a unique state ω_0 on the infinite lattice such that $\omega_0(\mathbb{1} - A_s) = \omega_0(\mathbb{1} - B_p) = 0$ for all stars s and plaquettes p of the infinite square lattice. AFH prove this using the algebra satisfied by the A_s and B_p, by showing that the vanishing of these expectations determines all expectation values.

In terms of random dual paths ω_0 looks like a gas of loops on (dual) $\mathbb{Z}^2$:



The probability of an infinite path (black edges) or dual path (red edges) vanishes.

However, it is also clear that the class of configurations that have one half-infinite dual path ending in p, is also stable under the action of the operators A_s:



The end point can be moved around by local operators but cannot be removed. These are excited states of energy 2.

Note that the equal-weight superposition of all such configurations with one end-point in a fixed plaquette, is an eigenstate of the Hamiltonian.

In fact, all configurations correspond to a configuration of dual paths, some open, some closed. Local operators can locally modify them by flipping spins.

Note that the role of σ^3 and σ^1 can be interchanged if we replace the lattice $\mathbb{Z}^2$ by the dual lattice, again $\mathbb{Z}^2$ (and the same set of spins). Thus, duality is actually a symmetry in this case, and the model is $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric.
(This symmetry is broken in the set of all infinite-volume ground states.)

The Gelfand-Naimark-Segal representation

States of the infinite system on all of $\mathbb{Z}^2$ are well-defined as functionals on the algebra of local observables. For each finite $\Lambda \subset \mathcal{E}(\mathbb{Z}^2)$, we consider the algebra $\mathcal{A}_\Lambda$ generated by $\{\sigma_x^\alpha \mid \alpha = 1, 2, 3,\ x \in \Lambda\}$.

$$\mathcal{A}_{\text{loc}} = \bigcup_{\Lambda \subset \mathbb{Z}^2} \mathcal{A}_\Lambda, \quad \mathcal{A} = \overline{\mathcal{A}_{\text{loc}}}.$$

A state is a positive linear functional ω on $\mathcal{A}$, normalized so that $\omega(\mathbb{1}) = 1$, that assigns to each observable its expectation.

The Gelfand-Naimark-Segal (GNS) representation assigns to each state ω a unique (up to unitary) triple $(\mathcal{H}, \pi, \Omega)$, where

- $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a representation with cyclic vector $\Omega \in \mathcal{H}$;
- $\omega(A) = \langle \Omega, \pi(A)\Omega \rangle$.

Consider ω_0, the unique frustration-free ground state of the Toric Code model on $\mathbb{Z}^2$, with GNS triple $(\mathcal{H}_0, \pi_0, \Omega_0)$.
How to define the state which differs from ω_0 by the presence of the end point of a half-infinite dual path?
Let ρ be a dual path beginning in p and ending in q, and define the unitary operator

$$F_\rho^\mu = \prod_{x \in \rho} \sigma_x^1,$$

We could try to define $\lim_{q \to \infty} \pi_0(F_\rho^\mu)\Omega_0$. But this is no good: it converges weakly to 0.

The solution is to use automorphisms of the form:

$$\tau_n^\mu(A) = F_{\rho_n}^\mu A F_{\rho_n}^\mu, \quad A \in \mathcal{A}_{\mathrm{loc}}.$$

τ_n^μ extends to an automorphism of the observable algebra and so does the limit $\tau^\mu(A) = \lim_{n \to \infty} \tau_n^\mu(A)$, if we take a sequence of dual lattice paths ρ_n, given by the first n edges in a given half-inifinite path.

Then, following (Naaijkens 2011), define

$$\omega_\mu = \omega_0 \circ \tau^\mu.$$

One can show that ω_μ only depends on the endpoint of the half-finite path, the path itself.

What is the GNS representation of ω_μ?

$$\begin{aligned}\omega_\mu(A) &= \omega_0 \circ \tau^\mu(A) = \langle \Omega_0, \pi_0(\tau^\mu(A))\Omega_0 \rangle \\ &= \lim_n \langle \pi_0(F^\mu_{\rho_n})\Omega_0, \pi_0(A)\pi_0(F^\mu_{\rho_n})\Omega_0 \rangle.\end{aligned}$$

Therefore, we can take the GNS triple given by $(\mathcal{H}_0, \pi_\mu, \Omega_0)$, with $\pi_\mu = \pi_0 \circ \tau^\mu$.

Inequivalent representations

The GNS rep of any state ω is unique up to unitary equivalence: if $(\mathcal{H}_j, \pi_j, \Omega_j)$, $j = 1, 2$, are two GNS triples for ω, there exists a unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $U\Omega_1 = \Omega_2$ and

$$\pi_2(A) = U\pi_1(A)U^*, \ \mathcal{A} \in \mathcal{A}. \tag{1}$$

When we have (1) but not necessarily $U\Omega_1 = \Omega_2$, we say that the representations π_1 and π_2 are equivalent. If π_1 and π_2 are not equivalent they are called inequivalent. Example: let $\Omega_1 \in \mathcal{H}_0$ be a unit vector in the GNS space of ω_0. If $\Omega_1 \neq \Omega_0$, the state

$$\omega_1(A) = \langle \Omega_1, \pi_0(A)\Omega_1 \rangle$$

is different from ω_0. $(\mathcal{H}_0, \pi_0, \Omega_1)$ is a GNS triple for ω_1. However, using any unitary U such that $\Omega_1 = U\Omega_0$, we find that $(\mathcal{H}_0, U^*\pi_0(\cdot)U, \Omega_0)$ also is a GNS triple for ω_1 and the representations π_0 and $\pi_1 = U^*\pi_0(\cdot)U$ are equivalent.

Example:
Let ρ be a finite dual path and F^{μ}_{ρ} the unitary operator

$$F^{\mu}_{\rho} = \prod_{x \in \rho} \sigma^{1}_{x}.$$

Then:

$$\begin{aligned} \omega_{\rho}(A) &= \omega_{0}(F^{\mu}_{\rho} A F^{\mu}_{\rho}) \\ &= \langle \Omega_{0}, \pi_{0}(F^{\mu}_{\rho} A F^{\mu}_{\rho}) \Omega_{0} \rangle \\ &= \langle \pi_{0}(F^{\mu}_{\rho}) \Omega_{0}, \pi_{0}(A) \pi_{0}(F^{\mu}_{\rho}) \Omega_{0} \rangle . \end{aligned}$$

$(\mathcal{H}_{0}, \pi_{0}, \Omega_{\rho} = \pi_{0}(F^{\mu}_{\rho}) \Omega_{0})$ is a GNS triple for ω_{ρ}. These representations are equivalent.

Example:
Recall the automorphisms τ^{μ} corresponding to a half-infinite path and the state

$$\omega_{\mu} = \omega_0 \circ \tau^{\mu}.$$

The GNS representation of ω_{μ} is given by $(\mathcal{H}_0, \pi_0 \circ \tau^{\mu}, \Omega_0)$:

$$\omega_{\mu}(A) = \omega_0 \circ \tau^{\mu}(A) = \langle \Omega_0, \pi_0(\tau^{\mu}(A))\Omega_0 \rangle,$$

but there does not exist a unitary U such that $\pi_0 \circ \tau^{\mu}(A) = U^* \pi_0(A) U$. The GNS representations of ω_0 and ω_{μ} are inequivalent.

Infinite volume ground states

A state ω on $\mathcal{A}$ is called a ground state of the infinite (Toric Code) model with finite-volume Hamiltonians H_Λ if

$$\lim_{\Lambda\uparrow\mathbb{Z}^2} \omega(A^*[H_\Lambda, A]) \geq 0, \text{ for all } A \in \mathcal{A}_{\rm loc}. \qquad (*)$$

The frustration-free ground state ω_0 discussed before satisfies this condition since we have

$$\omega_0(A^*[H_\Lambda, A]) = \omega_0(A^* H_\Lambda A) - \omega_0(A^* A H_\Lambda).$$

The first term is ≥ 0 and the second term vanishes by Cauchy-Schwarz and the frustration-free ground state property:

$$|\omega_0(A^* A H_\Lambda)|^2 \leq \omega_0((A^* A)^2)\omega_0(H_\Lambda^2) = 0.$$

It is also not hard to show that $\omega_\mu = \omega_0 \circ \tau_\mu$ also satisfies (*). Physically, this is an expression of the fact that the end point of the half-infinite string can be moved but not removed by local operators.

By the duality symmetry mentioned before, for each path along edges of the lattice (as opposed to the dual lattice) we define a unitary operator

$$F^e_\rho = \prod_{x \in \rho} \sigma^3_x,$$

and using these for a sequence of finite approximations of a half-infinite path we can define automorphisms τ^e, and states $\omega_e = \omega_0 \circ \tau^e$. These states depend only on the end-point of the path, and they also satisfy (*).

One can also compose an automorphism of the type τ^μ with one of the type τ^e and define states

$$\omega_{e\mu} = \omega \circ \tau^e \circ \tau^\mu.$$

Again, one can show $\omega_{e\mu}$ satisfies (*).

Since the end-point of a half-infinite path can be moved to any other position by composing with a unitary in $\mathcal{A}_{\mathrm{loc}}$, any two ω_α, with the same label $\alpha \in \{e, \mu, e\mu\}$, and different end points are unitarily equivalent and can be represented as vectors in the same Hilbert space, one can also make coherent superpositions of them, and those will again be infinite-volume ground states.
This gives us 4 classes of ground states:

$$K^0 = \{\omega_0\}, K^e, K^\mu, K^{e\mu},$$

which correspond to: the vacuum state ω_0, which is translation invariant, and the linear and convex combinations of each type: ω_e, ω_μ, and $\omega_{e\mu}$, respectively.
Within each class the states are unitarily equivalent, but states from different classes are inequivalent (Naaijkens 2011).

These four classes are all infinite-volume ground states in the sense of (*).

Theorem (Cha-Naaijkens-N, CMP 2018)

The set of all solution of (*) for the TCM is the closed convex hull of the states $\omega_0, \omega_e, \omega_\mu, \omega_{e\mu}$*, and all their translates and coherent superpositions of thereof.*
That is, there are 4 equivalence classes of ground states:

$$K^0 = \{\omega_0\}, K^e, K^\mu, K^{e\mu}.$$

The four classes of ground states are in one-to-one correspondence to the four superselection sectors obtained by Naaijkens.

What are superselection sectors?

In the framework of 'local quantum physics' (Doplicher-Haag-Roberts) a superselection sector is an equivalence class of representations of the observable algebra generated by composing certain types of endopmorphisms with the vacuum representation These equivalence classes are typically labeled by the values of one or more conserved quantities of the theory, called charges.

Let $\pi_0 : \mathcal{A} \to \mathcal{B}(\mathcal{H}_0)$ denote the GNS representation of the vacuum state ω_0. Then, for any endomorphism ρ of $\mathcal{A}$, $\pi_0 \circ \rho$ is another representation of $\mathcal{A}$, and we consider equivalence classed of such representations defined by unitary equivalence: $\pi_1 \sim \pi_0$ if there exists a unitary $U \in \mathcal{B}(\mathcal{H}_0)$ such that $\pi_1(A) = U^* \pi_0(A) U, A \in \mathcal{A}$.

Why superselection sectors? The importance of stability

With the Toric Code Model, and the more general class of quantum double models, Kitaev (Annals of Physics, 2003, 2006) demonstrated the existence of two-dimensional lattice systems with toplogically ordered ground states and anyonic (as opposed to fermionic or bosonic) particle-like excitations.

For anyons to be 'real', their existence should be robust (stable) under small changes of the Kitaev's Hamiltonians, which are toy models lacking common features of physical systems.

The structure of the space of ground states may or may not be stable but, under a set of physical conditions, the superselection sector structure is.

To obtain something physically meaningful we have restrict to ρ that have (at least) two essential properties:
- Almost-locality in cones: we denote the set of cones in $\mathbb{Z}^2$ with opening angle α by $\mathcal{C}_\alpha$ and require of ρ that there is $\alpha \in (0, \pi)$ and $\Lambda \in \mathcal{C}_\alpha$, such that for all $\epsilon \in (0, \pi - \alpha)$, $k \geq 0$

$$\lim_{n\to\infty} n^k \sup_{A\in\mathcal{A}_{\Lambda(\epsilon)^c - n}, \|A\|=1} \|\rho(A) - A\| = 0$$

where '$-n$' denotes translation by n in the direction opposite to the forward direction of the axis of Λ, $\Lambda(\epsilon)$ is the cone obtained from Λ by widening its opening angle by ϵ.
- transportability with respect to the vacuum state: for any two cones $\Lambda, \Lambda' \in \mathcal{C}_\alpha$, and ρ (almost) localized in Λ, there is an equivalent ρ' (almost) localized in Λ'.
'Almost locality' is the quasi-local version of the 'locality' employed by Doplicher-Haag-Roberts (1971-74) in algebraic QFT and the strict locality in cones used for the TCM by Naaijkens (2011). It is used in Cha's PhD thesis (2017) to treat perturbations of TCM.

Superselection sectors of the TCM

The superselection sectors of the TCM given as the equivalence classes of automorphisms localized in cones (Naaijkens 2011) is given by 4 classes of states equivalent to 4 classes of ground states $K^0, K^e, K^\mu, K^{e\mu}$ and can be given the structure of the braided C^* tensor category of the representations of the quantum double $\mathcal{D}(G = \mathbb{Z}_2)$.

If we add a finite-energy condition, we can show that this structure is stable under uniformly small perturbations of the TCM.

In particular, the same type of anyons describe its low-energy excitations.

Quasi-locality and automorphic equivalence - the spectral flow

Consider perturbations are of the form

$$H_\Lambda(s) = H_\Lambda^{TCM} + s \sum_{X \subset \Lambda} \Phi(X).$$

with Φ an interaction such that for some $a > 0$

$$\|\Phi\|_a = \sup_{x,y \in \mathbb{Z}^2} e^{a|x-y|} \sum_{\substack{X \subset \mathbb{Z}^2 \\ x,y \in X}} \|\Phi(X)\| < \infty,$$

For what follows it will be important that H_Λ^{TCM} is frustration-free, gapped, and that its ground states satisfy a property called Local Topological Quantum Order.

Stability of the superselection sectors

Theorem (Cha 2017, Cha-Naaijkens-N (in preparation))

There exists $s_0 > 0$ such that for $|s| \leq s_0$, there exists an automorphism α_s with the following properties:
(i) α_s is the dynamics corresponding to a time-dependent short-range interaction $\Psi(s)$
(ii) $\omega_0 \circ \alpha_s$ is a translation invariant infinite volume ground states of the perturbed model, with a positive spectral gap;
(iii) $K^k \circ \alpha_s$, for $k \in \{0, e, \mu, e\mu\}$, describe the finite-energy superselection sectors of the perturbed model and are generated by almost localized automorphisms $\tau_s^k = \alpha_s^{-1} \circ \tau^k \circ \alpha_s$;
(iv) The set of superselection sectors of the perturbed model has the same braided (C^*-) fusion tensor category structure as TCM.

The main tool in the proof, the automorphisms α_s, called the spectral flow or Hastings' evolution, are constructed using Lieb-Robinson bounds Bachmann-Michalakis-N-Sims (2012). Review paper(s) N-Sims-Young, in prep.
Quasi-locality properties follow from the fact that α_s is the dynamics corresponding to a time-dependent short-range interaction $\Psi(s)$ with, for some $c > 0$,

$$\sup_{x,y\in\mathbb{Z}^2} e^{c|x-y|/(\log|x-y|)^2} \sum_{\substack{X\subset\mathbb{Z}^2\\ x,y\in X}} \sup_{|s|\leq s_0} \|\Psi(s,X)\| < \infty.$$

Lieb-Robinson bounds are used to prove this and, in turn, this structure implies that α_s satisfies a Lieb-Robinson bound, which is used to prove that $\tau_s = \alpha_s^{-1} \circ \tau \circ \alpha_s$ is almost localized in a cone whenever τ is.
To define the fusion and braiding structure we use the framework of (bi-)asymptopias of Buchholz-Doplicher-Morchio-Roberts-Strocchi, 2007.

Comments and Outlook

- Exploiting quasi-locality is an essential ingredient in many recent results, and can be applied to extended operators.
- Frustration-free models turn out to be a very useful class of examples.
- Stability of the superselection sectors also comes with stability of anyons (fusion and braiding). Anyons exist.
- We need better techniques to prove spectral gaps in 2 and more dimensions (Lemm-Mozgunov, 2018).
- The nature and role ‘edges states’ for infinite systems with boundary needs mathematical investigation. Extension to non-abelian anyons.