Lecture 2: Asymptotic Analysis of integrals, and a Mathematica Code

Lecture plan. We will start by carrying out an asymptotic analysis of the integral (from the previous lecture)

$$u(x,t) = \int_{\mathbb{R}} \hat{f}(k) e^{ikx+ik^3 t} dk. \tag{1}$$

We will see that as $t \to \infty$, the solution may be described as a superposition of waves, dispersing according to a so-called "dispersion relation".

Following that we will investigate Peter Miller's code "Solitons.nb" which contains a simple algorithm for solving certain nonlinear pdes numerically.

Asymptotic analysis of integrals

The solution $u(x,t)$ can be re-written (as we did last time):

$$u(x,t) = \int_{\mathbb{R}} \hat{f}(k) e^{it\phi(k)} dk, \quad \phi(k) = k^3 + \left(\frac{x}{t}\right) k. \tag{2}$$

The function $\phi(k)$ possesses two critical points, called $k_{\pm}$. We will for starters suppose that $x < 0$ and $t > 0$ so that the two critical points are both real. In addition, we shall assume that as $t \to \infty$, the ratio x/t converges to a negative number ξ.

Let I_+ denote an interval of length 2δ surrounding the point k_+, and similarly with I_- and k_-. The integral may be broken up into 5 pieces, of which two are integrals over I_+ and I_-. Let's consider the contribution from the interval I_-. Since we are in a vicinity of the critical point k_+, it is possible to approximate the integral as follows:

$$\int_{I_+} \hat{f}(k) e^{it\phi(k)} dk \approx \int_{I_+} \hat{f}(k_+) e^{it\left(\phi(k_+) + \frac{1}{2}\phi''(k_+)(k-k_+)^2\right)} dk. \tag{3}$$

It is a major piece of work to justify this approximation. Skipping that for the moment, this final integral itself can be approximated:

$$\begin{aligned} \int_{I_+} \hat{f}(k_+) e^{it\left(\phi(k_+) + \frac{1}{2}\phi''(k_+)(k-k_+)^2\right)} dk \quad &\approx \quad \hat{f}(k_+) e^{it\phi(k_+)} \int_{-\infty}^{\infty} e^{\frac{it}{2}\phi''(k_+)(k-k_+)^2} dk \\ &= \hat{f}(k_+) e^{it\phi(k_+)} e^{i\pi/4} \sqrt{\frac{2\pi}{t\ \phi''(k_+)}} \end{aligned} \tag{4}$$

The detailed justification of these steps will be completed in the lecture.

One may similarly estimate the contribution from I_-:

$$\int_{I_-} \hat{f}(k) e^{it\phi(k)} dk \approx \hat{f}(k_-) e^{it\phi(k_-)} e^{-i\pi/4} \sqrt{\frac{2\pi}{-t\ \phi''(k_-)}}\ . \tag{5}$$

The contribution from the three remaining subintervals of $\mathbb{R}$ are controlled by integration by parts arguments. For example, consider the contribution from $I_3 = (k_+ + \delta, \infty)$:

$$\int_{I_3} \hat{f}(k) e^{it\phi(k)} dk = \int_{I_3} \frac{\hat{f}(k)}{it\phi'(k)} \frac{d}{dk} e^{it\phi(k)} dk = \int_{I_3} \frac{d}{dk} \frac{\hat{f}(k)}{it\phi'(k)} e^{it\phi(k)} dk - \int_{I_3} \left(\frac{\hat{f}(k)}{it\phi'(k)}\right)' e^{it\phi(k)} dk \tag{6}$$

$$= -\frac{\hat{f}}{it\phi'} e^{it\phi}\Bigg|_{k_+ + \delta} - \int_{I_3} \left(\frac{\hat{f}(k)}{it\phi'(k)}\right)' e^{it\phi(k)} dk. \tag{7}$$

Now the first term in (7) is clearly bounded by ct^{-1}, and so is the second term:

$$\left| \int_{I_3} \left(\frac{\hat{f}(k)}{it\phi'(k)}\right)' e^{it\phi(k)} dk \right| \leq \frac{c}{t} \tag{8}$$

(provided we assume that $\hat{f}$ and $\hat{f}'$ are integrable.)

Combining all of this, we have established the following:

$$u(x,t) \approx \hat{f}(k_+)e^{it\phi(k_+)}e^{i\pi/4}\sqrt{\frac{2\pi}{t\ \phi''(k_+)}} + \hat{f}(k_-)e^{it\phi(k_-)}e^{-i\pi/4}\sqrt{\frac{2\pi}{-t\ \phi''(k_-)}} + \mathcal{O}\left(t^{-1}\right)\ . \tag{9}$$

Now the two dominant terms can be interpreted as traveling waves. To see this, let's set

$$x = -\xi\tau - X, \quad t = \tau + T. \tag{10}$$

In these new coordinates, we find that

$$e^{it\phi(k_\pm)} = e^{\mp\frac{2it\xi^{3/2}}{3^{3/2}}}e^{\mp i\sqrt{\frac{\xi}{3}}(X-\xi T)} \tag{11}$$

Now the interpretation is as follows: the above represents a wave packet traveling with velocity ξ. The frequency of this wave packet is $\sqrt{\xi/3}$. So for large t, the solution $u(x,t)$ can be described as follows: it decomposes into packets of waves, whose frequency and velocity are related as follows:

$$\text{group velocity} = 3 \times (\text{frequency})^2\,. \tag{12}$$

This is often written as a dispersion relation, in the form

$$w(k) = 3k^2. \tag{13}$$

You might ask how much energy resides at a given frequency range. This can be determined from the prefactors in formula (9) - it is determined from the initial data through $\hat{f}(k_\pm)$.