Lecture 2: Asymptotic Analysis of Integrals, and a Mathematica Code

Lecture plan. We will start by carrying out an asymptotic analysis of the integral (from the previous lecture)

$$u(x, t) = \int_{\mathbb{R}} \hat{f}(k)e^{ikx + itk^3}dk.$$  \hspace{1em} (1)

We will see that as $t \to \infty$, the solution may be described as a superposition of waves, dispersing according to a so-called “dispersion relation”.

Following that we will investigate Peter Miller’s code “Solitons.nb” which contains a simple algorithm for solving certain nonlinear pdes numerically.

Asymptotic analysis of integrals

The solution $u(x, t)$ can be re-written (as we did last time):

$$u(x, t) = \int_{\mathbb{R}} \hat{f}(k)e^{it\phi(k)}dk, \quad \phi(k) = k^3 + \left(\frac{x}{t}\right)k.$$  \hspace{1em} (2)

The function $\phi(k)$ possesses two critical points, called $k_{\pm}$. We will for starters suppose that $x < 0$ and $t > 0$ so that the two critical points are both real. In addition, we shall assume that as $t \to \infty$, the ratio $x/t$ converges to a negative number $\xi$.

Let $I_{+}$ denote an interval of length 25 surrounding the point $k_{+}$, and similarly with $I_{-}$ and $k_{-}$. The integral may be broken up into 5 pieces, of which two are integrals over $I_{+}$ and $I_{-}$. Let’s consider the contribution from the interval $I_{-}$. Since we are in a vicinity of the critical point $k_{+}$, it is possible to approximate the integral as follows:

$$\int_{I_{-}} \hat{f}(k)e^{it\phi(k)}dk \approx \int_{I_{+}} \hat{f}(k)e^{it\phi(k)}dk.$$  \hspace{1em} (3)

It is a major piece of work to justify this approximation. Skipping that for the moment, this final integral itself can be approximated:

$$\int_{I_{+}} \hat{f}(k)e^{it\phi(k)}dk \approx \hat{f}(k_{+})e^{it\phi(k_{+})} \int_{-\infty}^{\infty} e^{\frac{it}{2}\phi''(k_{+})(k-k_{+})^2}dk = \hat{f}(k_{+})e^{it\phi(k_{+})}e^{i\pi/4} \sqrt{\frac{2\pi}{t\phi''(k_{+})}}.$$  \hspace{1em} (4)

The detailed justification of these steps will be completed in the lecture.

One may similarly estimate the contribution from $I_{-}$:

$$\int_{I_{-}} \hat{f}(k)e^{it\phi(k)}dk \approx \hat{f}(k_{-})e^{it\phi(k_{-})}e^{-i\pi/4} \sqrt{\frac{2\pi}{-t\phi''(k_{-})}}.$$  \hspace{1em} (5)

The contribution from the three remaining subintervals of $\mathbb{R}$ are controlled by integration by parts arguments. For example, consider the contribution from $I_{3} = (k_{+} + \delta, \infty)$:

$$\int_{I_{3}} \hat{f}(k)e^{it\phi(k)}dk = \int_{I_{3}} \frac{\hat{f}(k)}{it\phi'(k)} \frac{d}{dk}e^{it\phi(k)}dk = \int_{I_{3}} \frac{d}{dk} \left( \hat{f}(k) \right) \frac{1}{it\phi'(k)} e^{it\phi(k)}dk - \int_{I_{3}} \left( \hat{f}(k) \right) \left( \frac{d}{dk} \frac{1}{it\phi'(k)} \right) e^{it\phi(k)}dk.$$  \hspace{1em} (6)

$$= -\frac{\hat{f}(k)}{it\phi'} e^{it\phi} \bigg|_{k_{+}+\delta} - \int_{I_{3}} \left( \hat{f}(k) \right) \left( \frac{d}{dk} \frac{1}{it\phi'(k)} \right) e^{it\phi(k)}dk.$$  \hspace{1em} (7)

Now the first term in (7) is clearly bounded by $ct^{-1}$, and so is the second term:

$$\left| \int_{I_{3}} \left( \hat{f}(k) \right) \left( \frac{d}{dk} \frac{1}{it\phi'(k)} \right) e^{it\phi(k)}dk \right| \leq \frac{c}{t}$$  \hspace{1em} (8)

(provided we assume that $\hat{f}$ and $\hat{f}'$ are integrable. )
Combining all of this, we have established the following:

\[
\begin{align*}
\tag{9} u(x, t) & \approx \hat{f}(k_+) e^{it\phi(k_+)} e^{i\pi/4} \sqrt{\frac{2\pi}{t \phi''(k_+)}} + \hat{f}(k_-) e^{it\phi(k_-)} e^{-i\pi/4} \sqrt{\frac{2\pi}{-i \phi''(k_-)}} + O \left( t^{-1} \right). \\
\end{align*}
\]

Now the two dominant terms can be interpreted as traveling waves. To see this, let’s set

\[
\begin{align}
\tag{10} x &= -\xi \tau - X, \quad t = \tau + T.
\end{align}
\]

In these new coordinates, we find that

\[
\begin{align}
\tag{11} e^{it\phi(k_\pm)} &= e^{i \frac{2\xi(k_\pm)^{3/2}}{3} t} e^{\mp i \sqrt{\xi/3}} (X - \xi T).
\end{align}
\]

Now the interpretation is as follows: the above represents a wave packet traveling with velocity $\xi$. The frequency of this wave packet is $\sqrt{\xi/3}$. So for large $t$, the solution $u(x, t)$ can be described as follows: it decomposes into packets of waves, whose frequency and velocity are related as follows:

\[
\tag{12} \text{group velocity} = 3 \times \text{(frequency)}^2.
\]

This is often written as a dispersion relation, in the form

\[
\tag{13} w(k) = 3k^2.
\]

You might ask how much energy resides at a given frequency range. This can be determined from the prefactors in formula (9) - it is determined from the initial data through $\hat{f}(k_\pm)$.