Lecture 3: Completing the details of asymptotic analysis of an integral, a Mathematica Code

Lecture plan. We will complete the details of the asymptotic analysis of the integral from last time,

$$u(x,t) = \int_{\mathbb{R}} \hat{f}(k) e^{ikx+ik^3t} dk. \tag{1}$$

Following that we will investigate Peter Miller's code "Solitons.nb" which contains a simple algorithm for solving certain nonlinear pdes numerically.

Asymptotic analysis of integrals

The solution $u(x,t)$ can be re-written (as we did last time):

$$u(x,t) = \int_{\mathbb{R}} \hat{f}(k) e^{it\phi(k)} dk, \quad \phi(k) = k^3 + \left(\frac{x}{t}\right) k. \tag{2}$$

The function $\phi(k)$ possesses two critical points, called $k_\pm$. Assumptions from Lecture 2 are as follows:

1. The initial data $f(x)$ is such that $\hat{f}(k)$ and $\hat{f}'$ are both in $\mathrm{L}^1(\mathbb{R})$.
2. There are positive constants C and D so that as $t \to \infty$, the ratio x/t obeys

$$C < -\frac{x}{t} < D \tag{3}$$

We then presented arguments which led us to conclude that

$$u(x,t) \approx \hat{f}(k_+) e^{it\phi(k_+)} e^{i\pi/4} \sqrt{\frac{2\pi}{t\ \phi''(k_+)}} + \hat{f}(k_-) e^{it\phi(k_-)} e^{-i\pi/4} \sqrt{\frac{2\pi}{-t\ \phi''(k_-)}} + \mathcal{O}\left(t^{-1}\right) \ . \tag{4}$$

Remark on notation: The term $\mathcal{O}(t^{-1})$ has the following interpretation: if one writes "$h(t) = \mathcal{O}(t^{-1})$ as $t \to \infty$" then this means that there is T sufficiently large, a positive constant c, so that for all $t > T$, one has $|h(t)| \leq ct^{-1}$.

In Lecture 2 we left to be proven the following approximations:

$$\int_{I_+} \hat{f}(k) e^{it\phi(k)} dk \approx \int_{I_+} \hat{f}(k_+) e^{it\left(\phi(k_+) + \frac{1}{2}\phi''(k_+)(k-k_+)^2\right)} dk, \tag{5}$$

and

$$\begin{aligned} \int_{I_+} \hat{f}(k_+) e^{it\left(\phi(k_+) + \frac{1}{2}\phi''(k_+)(k-k_+)^2\right)} dk \quad &\approx \quad \hat{f}(k_+) e^{it\phi(k_+)} \int_{-\infty}^{\infty} e^{\frac{it}{2}\phi''(k_+)(k-k_+)^2} dk \\ &= \hat{f}(k_+) e^{it\phi(k_+)} e^{i\pi/4} \sqrt{\frac{2\pi}{t\ \phi''(k_+)}} \end{aligned} \tag{6}$$

Method 1 One seeks a transformation $k \mapsto \lambda$ so that for all $k \in I_+$,

$$\phi(k) = \phi(k_+) + \lambda^2. \tag{7}$$

Then the integral may be rewritten as follows:

$$\int_{I_+} \hat{f}(k) e^{it\phi(k)} dk = \int_{\lambda(I_+)} \hat{f}(k(\lambda)) k'(\lambda) e^{it\lambda^2} d\lambda = \int_{\lambda(I_+)} h(\lambda) e^{it\lambda^2} d\lambda, \tag{8}$$

where $h(\lambda) = \hat{f}(\lambda) k'(\lambda)$. Next one pulls off the behavior at $\lambda = 0$:

$$\int_{I_+} \hat{f}(k) e^{it\phi(k)} dk = \int_{\lambda(I_+)} h(0) e^{it\lambda^2} d\lambda + \int_{\lambda(I_+)} \left(h(\lambda) - h(0)\right) e^{it\lambda^2} d\lambda. \tag{9}$$

The first term can be estimated by complex variables trickery we saw in Lecture 2, and one can prove:

$$\int_{\lambda(I_+)} h(0) e^{it\lambda^2} d\lambda = h(0) \sqrt{\frac{\pi}{t}} \left(1 + \mathcal{O}\left(t^{-1/2}\right)\right) \tag{10}$$

and the second term requires more manipulation. For the second term we can re-write it as follows:

$$\int_{\lambda(I_+)} (h(\lambda) - h(0))\, e^{it\lambda^2} d\lambda = \int_{\lambda(I_+)} \left(\frac{h(\lambda) - h(0)}{2it\lambda} \right) \frac{d}{d\lambda} e^{it\lambda^2} d\lambda \tag{11}$$

and now we may integrate by parts:

$$\int_{\lambda(I_+)} (h(\lambda) - h(0))\, e^{it\lambda^2} d\lambda = \left(\frac{h(\lambda) - h(0)}{2it\lambda} \right) e^{it\lambda^2} \Big|_{\text{endpoints}} - \int_{\lambda(I_+)} \left(\frac{h(\lambda) - h(0)}{2it\lambda} \right)' e^{it\lambda^2} d\lambda\ . \tag{12}$$

Both terms in this last equation may be bounded by ct^{-1}, provided $\left(\frac{h(\lambda)-h(0)}{2it\lambda} \right)$ has a bounded derivative. This turns out to be an additional assumption on the function $\hat{f}(k)$, which is now required to have a bounded second derivative for k near 0.

Method 2 Another method which is quite different is as follows. Let's suppose that there exists a function $\Phi(u, v)$ so that

(c1) For $u \in \mathbb{I}_+$, $\Phi(u, 0) = \phi(u)$.
(c2) $\Phi(u, u) = \phi(k_+) + \frac{1}{2}\phi''(k_+)\,(u + iv - k_+)^2$
(c3) $|(\partial_x + i\partial_y)\,\Phi(u, v)| \leq c(u - k_+)v$ for all (u, v) near I_+,
(c4) $\mathrm{Im}(\Phi(u, v)) \geq c(u - k_+)v$ for all (u, v) near I_+,

where c is a positive constant.

Now we can re-write the integral using Stokes' theorem (or Green's theorem, if you choose):

$$\begin{aligned} \int_{I_+} \hat{f}(k) e^{it\phi(k)} dk = -\int_\Gamma \hat{f}(u) e^{it\Phi(u,v)} d(u + iv) + \\ +2i \iint_{\mathcal{D}_+} \frac{1}{2}(\partial_u + i\partial_v)\, \hat{f}(u) e^{it\Phi(u,v)} dudv - 2i \iint_{\mathcal{D}_-} \frac{1}{2}(\partial_u + i\partial_v)\, \hat{f}(u) e^{it\Phi(u,v)} dudv, \end{aligned} \tag{13}$$

where Γ is the contour comprised of a vertical segment from $k_+ + \delta$ to $k_+ + \delta + i\delta$, followed by the line segment connecting $k_+ + \delta + i\delta$ to $k_+ - \delta - i\delta$, and ending with the vertical line segment from $k_+ - \delta - i\delta$ to $k_+ - \delta$, and $\mathcal{D}$ denotes the interior of the two triangles formed by this contour and the real axis.

The contribution to the first integral from the diagonal line segment, using (c2), is basically a Gaussian integral:

$$-\int_\Gamma \hat{f}(u) e^{it\Phi(u,v)} d(u + iv) = e^{i\pi/4} \int_{-\delta}^{\delta} \hat{f}(k_+ + s\sqrt{2}/2) e^{-\frac{t\phi''(k_+)}{2}s^2} ds - \int_\uparrow \hat{f}(u) e^{it\Phi(u,v)} d(u + iv). \tag{14}$$

Now the first integral in (14) can be calculated (up to an explicitly controlled error term) and there remains the tasks of estimating

$$\int_\uparrow \hat{f}(u) e^{it\Phi(u,v)} d(u + iv) \tag{15}$$

and

$$2i \iint_{\mathcal{D}_+} \frac{1}{2}(\partial_u + i\partial_v)\, \hat{f}(u) e^{it\Phi(u,v)} dudv - 2i \iint_{\mathcal{D}_-} \frac{1}{2}(\partial_u + i\partial_v)\, \hat{f}(u) e^{it\Phi(u,v)} dudv \tag{16}$$

and both of these will be estimated in class, using (c3) and (c4).

There remains the question: *Can we prove that the function $\Phi(u, v)$ exists?* The answer is yes, and the details of its existence will be carried out in class.

In conclusion, let us recall from Lecture 2 that the two dominant terms can be interpreted as traveling waves. We set

$$x = -\xi\tau - X, \quad t = \tau + T. \tag{17}$$

In these new coordinates, we find that

$$e^{it\phi(k_\pm)} = e^{\mp\frac{2it\xi^{3/2}}{3^{3/2}}} e^{\mp i\sqrt{\frac{\xi}{3}}(X - \xi T)} \tag{18}$$

Now the interpretation is as follows: the above represents a wave packet traveling with velocity ξ. The frequency of this wave packet is $\sqrt{\xi/3}$. So for large t, the solution $u(x,t)$ can be described as follows: it decomposes into packets of waves, whose frequency and velocity are related as follows:

$$\text{group velocity} = 3 \times (\text{frequency})^2 . \tag{19}$$

This is often written as a dispersion relation, in the form

$$w(k) = 3k^2. \tag{20}$$

You might ask how much energy resides at a given frequency range. This can be determined from the prefactors in formula (4) - it is determined from the initial data through $\hat{f}(k_\pm)$.