LECTURE 3: COMPLETING THE DETAILS OF ASYMPTOTIC ANALYSIS OF AN INTEGRAL, A MATHEMATICA CODE

Lecture plan. We will complete the details of the asymptotic analysis of the integral from last time,

\[ u(x, t) = \int_{\mathbb{R}} \hat{f}(k)e^{itx + ik^3t}dk. \]

Following that we will investigate Peter Miller’s code “Solitons.nb” which contains a simple algorithm for solving certain nonlinear pdes numerically.

ASYMPTOTIC ANALYSIS OF INTEGRALS

The solution \( u(x, t) \) can be re-written (as we did last time):

\[ \int \]

(2) \[ u(x, t) = \int_{\mathbb{R}} \hat{f}(k)e^{it\phi(k)}dk, \quad \phi(k) = k^3 + \left(\frac{x}{t}\right)k. \]

The function \( \phi(k) \) possesses two critical points, called \( k_{\pm} \). Assumptions from Lecture 2 are as follows:

1. The initial data \( f(x) \) is such that \( \hat{f}(k) \) and \( \hat{f}' \) are both in \( L^1(\mathbb{R}) \).
2. There are positive constants \( C \) and \( D \) so that as \( t \to \infty \), the ratio \( x/t \) obeys

\[ C < -\frac{x}{t} < D \]

We then presented arguments which led us to conclude that

(3) \[ u(x, t) \approx \hat{f}(k_+ + \phi(k_+) + \frac{1}{t}\phi''(k_+)(k-k_+)^2)dk, \]

\[ \int \]

and

(6) \[ \int_{I_+} \hat{f}(k_+ + \phi(k_+) + \frac{1}{t}\phi''(k_+)(k-k_+)^2)dk = \approx \int_{I_+} \hat{f}(k_+ + \phi(k_+)) \int_{-\infty}^{\infty} e^{\frac{2\pi}{t}\phi''(k_+)(k-k_+)^2}dk \]

Method 1 One seeks a transformation \( k \mapsto \lambda \) so that for all \( k \in I_+ \),

\[ \phi(k) = \phi(k_+) + \lambda^2. \]

Then the integral may be rewritten as follows:

(8) \[ \int_{I_+} \hat{f}(k)e^{it\phi(k)}dk = \int_{\lambda(I_+)} \hat{f}(k(\lambda))k'(\lambda)e^{it\lambda^2}d\lambda = \int_{\lambda(I_+)} h(\lambda)e^{it\lambda^2}d\lambda, \]

where \( h(\lambda) = \hat{f}(\lambda)k'(\lambda) \). Next one pulls off the behavior at \( \lambda = 0 \):

(9) \[ \int_{I_+} \hat{f}(k)e^{it\phi(k)}dk = \int_{\lambda(I_+)} h(0)e^{it\lambda^2}d\lambda + \int_{\lambda(I_+)} (h(\lambda) - h(0)) e^{it\lambda^2}d\lambda. \]

The first term can be estimated by complex variables trickery we saw in Lecture 2, and one can prove:

(10) \[ \int_{\lambda(I_+)} h(0)e^{it\lambda^2}d\lambda = h(0)\sqrt{\frac{\pi}{t}}\left(1 + \mathcal{O}\left(t^{-1/2}\right)\right) \]
and the second term requires more manipulation. For the second term we can re-write it as follows:

\[
(11) \quad \int_{\lambda(I_+)} (h(\lambda) - h(0)) e^{it\lambda^2} d\lambda = \int_{\lambda(I_+)} \left( \frac{h(\lambda) - h(0)}{2it\lambda} \right) \frac{d}{d\lambda} e^{it\lambda^2} d\lambda
\]

and now we may integrate by parts:

\[
(12) \quad \int_{\lambda(I_+)} (h(\lambda) - h(0)) e^{it\lambda^2} d\lambda = \left. \left( \frac{h(\lambda) - h(0)}{2it\lambda} \right) e^{it\lambda^2} \right|_{\text{endpoints}} - \int_{\lambda(I_+)} \left( \frac{h(\lambda) - h(0)}{2it\lambda} \right)' e^{it\lambda^2} d\lambda.
\]

Both terms in this last equation may be bounded by $ct^{-1}$, provided \( \left( \frac{h(\lambda) - h(0)}{2it\lambda} \right) \) has a bounded derivative.

This turns out to be an additional assumption on the function \( \hat{f}(k) \), which is now required to have a bounded second derivative for \( k \) near 0.

**Method 2** Another method which is quite different is as follows. Let’s suppose that there exists a function \( \Phi(u,v) \) so that

(c1) For \( u \in \mathbb{R}_+ \), \( \Phi(u,0) = \phi(u) \).

(c2) \( \Phi(u,v) = \phi(k_+ + \frac{1}{2} \phi''(k_+)(u + iv - k_+)^2 \)

(c3) \( \left| (\partial_x + i\partial_y) \Phi(u,v) \right| \leq c(u - k_+)v \) for all \( (u,v) \) near \( I_+ \),

(c4) \( \text{Im}(\Phi(u,v)) \geq c(u - k_+)v \) for all \( (u,v) \) near \( I_+ \),

where \( c \) is a positive constant.

Now we can re-write the integral using Stokes’ theorem (or Green’s theorem, if you choose):

\[
(13) \quad \int_{I_+} \hat{f}(k)e^{it\phi(k)} dk = -\int_{\Gamma} \hat{f}(u)e^{it\Phi(u,v)} d(u + iv) + 2i \int_{\mathbb{D}_+} \frac{1}{2} (\partial_u + i\partial_v) \hat{f}(u)e^{it\Phi(u,v)} dudv - 2i \int_{\mathbb{D}_-} \frac{1}{2} (\partial_u + i\partial_v) \hat{f}(u)e^{it\Phi(u,v)} dudv,
\]

where \( \Gamma \) is the contour comprised of a vertical segment from \( k_+ + \delta \) to \( k_+ + \delta + i\delta \), followed by the line segment connecting \( k_+ + \delta + i\delta \) to \( k_+ - \delta - i\delta \), and ending with the vertical line segment from \( k_+ - \delta - i\delta \) to \( k_+ - \delta \), and \( \mathbb{D} \) denotes the interior of the two triangles formed by this contour and the real axis.

The contribution to the first integral from the diagonal line segment, using (c2), is basically a Gaussian integral:

\[
(14) \quad -\int_{\Gamma} \hat{f}(u)e^{it\Phi(u,v)} d(u + iv) = e^{i\pi/4} \int_{-\delta}^{\delta} \hat{f}(k_+ + s\sqrt{2}/2)e^{-\frac{s^2}{2}} ds - \int_{-\delta}^{\delta} \hat{f}(u)e^{it\Phi(u,v)} d(u + iv).
\]

Now the first integral in (14) can be calculated (up to an explicitly controlled error term) and there remains the tasks of estimating

\[
(15) \quad \int_{\Gamma} \hat{f}(u)e^{it\Phi(u,v)} d(u + iv)
\]

and

\[
(16) \quad 2i \int_{\mathbb{D}_+} \frac{1}{2} (\partial_u + i\partial_v) \hat{f}(u)e^{it\Phi(u,v)} dudv - 2i \int_{\mathbb{D}_-} \frac{1}{2} (\partial_u + i\partial_v) \hat{f}(u)e^{it\Phi(u,v)} dudv
\]

and both of these will be estimated in class, using (c3) and (c4).

There remains the question: *Can we prove that the function \( \Phi(u,v) \) exists?* The answer is yes, and the details of its existence will be carried out in class.

In conclusion, let us recall from Lecture 2 that the two dominant terms can be interpreted as traveling waves. We set

\[
(17) \quad x = -\xi t - X, \quad t = \tau + T.
\]

In these new coordinates, we find that

\[
(18) \quad e^{it\phi(k_\pm)} = e^{\frac{2iu\lambda^{3/2}}{3v\lambda}} e^{\mp i\sqrt{\frac{2}{3}}(X - \xi T)}
\]
Now the interpretation is as follows: the above represents a wave packet traveling with velocity $\xi$. The frequency of this wave packet is $\sqrt{\xi/3}$. So for large $t$, the solution $u(x,t)$ can be described as follows: it decomposes into packets of waves, whose frequency and velocity are related as follows:

$$\text{group velocity} = 3 \times (\text{frequency})^2.$$  

(19)

This is often written as a dispersion relation, in the form

$$w(k) = 3k^2.$$  

(20)

You might ask how much energy resides at a given frequency range. This can be determined from the prefactors in formula (4) - it is determined from the initial data through $\hat{f}(k_{\pm})$. 
