Lecture 4: The KdV equation: Mathematical structure and the discovery of Gardner, Green, Kruskal and Miura

Lecture plan. We will begin by summarizing the results of the asymptotic analysis of the integral,

$$u(x,t) = \int_{\mathbb{R}} \hat{f}(k) e^{ikx+ik^3 t} dk. \tag{1}$$

Then we will discuss a few exercises intended to broaden our understanding of the asymptotic analysis of integrals.

Following that we will begin a discussion of the KdV equation and the Schrödinger equation, and the amazing connection between these two equations.

The KdV equation:

$$u_t + uu_x + u_{xxx} = 0, \tag{2}$$

and the Schrödinger equation:

$$-6\psi_{xx} - u\psi = E\psi \tag{3}$$

A few asymptotic analysis of integral examples

A. The first example is the Gamma function (as mentioned in Lecture 1),

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \tag{4}$$

has the great property that if n is an integer, then $(n)! = \Gamma(n+1)$. The behavior of $n!$ for n large can be deduced by applying the same method. The answer is:

$$n! = \Gamma(n+1) = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} + \mathcal{O}\left(n^{-3}\right)\right). \tag{5}$$

See if you can establish this (or at least the leading order version).

B. A second example is the solution of Airy's equation,

$$y'' - xy = 0. \tag{6}$$

Seek a solution of the form $y(x) = \int_\gamma e^{ikx} f(k) dk$. Once you have determined the function $f(k)$ that works, specify valid contours. Once you have taken a valid contour, consider the question: what is the asymptotic behavior of the solution you have identified for $x \to \pm\infty$? *Here are some hints:*

  (1) First step: plug the above guess into the equation, and integrate by parts to remove the term $x \int_\gamma e^{ikx} f(k) dk$. (To do this, you will need to assume that the boundary terms do not contribute; this will force you to pick your contours γ later.) Then you should have an equation of the form

$$\int_\gamma e^{ikx}\left[(ik)^2 f(k) - i f'(k)\right] dk = 0. \tag{7}$$

  (2) In order for this to hold for all x, it follows that *the integrand must vanish.* Can you solve this ODE?

  (3) Third step: once you have determined $f(k)$, you must specify a contour γ for which the integral is convergent and your integration by parts is valid.

  (4) Fourth step: seek "stationary phase points" in the plane and carry out a "steepest descent analysis".

C. For the linear KdV equation, we have established asymptotics assuming that $C < -x/t < D$. What is the asymptotic behavior for $t \to \infty$, but with $0 < x/t < C$? More generally, you could imagine a complete asymptotic description in which asymptotics are given in regions which possess nontrivial overlaps. For example, there might be 5 regions:

  (1) **Region 1**: $\tilde{D} \leq -x/t < \infty$
  (2) **Region 2**: $\tilde{C} \leq -x/t \leq \hat{D}$ (with $\hat{D} > \tilde{D}$)
  (3) **Region 3**: $-\hat{C} \leq x/t \leq C_0$ (with $\hat{C} > \tilde{C}$)

(4) **Region 4**: $C_1 \leq x/t \leq D_0$ (with $C_1 < \hat{C}$)
(5) **Region 5**: $D_1 \leq -x/t < \infty$ (with $D_1 < D_0$).

THE VISCOUS BURGERS' EQUATION

Here I will briefly paraphrase Peter Miller's Lecture 4, where he discusses the viscous Burgers' equation,

$$u_t + uu_x - 3u_{xx} = 0, \tag{8}$$

although nonlinear, can be solved by an amazing linearizing transformation: the Cole-Hopf transform. One sets

$$u = -6\frac{\psi_x}{\psi} \tag{9}$$

and arrives at

$$\psi_t - 3\psi_{xx} = 0, \tag{10}$$

which is the (linear!) heat equation. This one can be solved by Fourier theory!

Another calculation that was mentioned in class, which you are encouraged to try: Consider the viscous Burgers' equation with small coefficient of diffusion (ν):

$$u_t + uu_x - \nu u_{xx} = 0, \tag{11}$$

$$u(x,0) = f(x) \tag{12}$$

where $\nu > 0$ is assumed to be small. In fact this problem challenges you to consider the limit $\nu \to 0$. Carry out the analogous solution procedure, adapted to the case of general $\nu > 0$. So you'll have a representation for the solution, which will depend on the initial data $f(x)$, and the parameter ν. Using your solution, study the behavior of the solution as $\nu \to 0$.

Hint: The Fourier representation of $\psi(x,t)$ (i.e. as an integral over k-space) will prove problematic (hard to prove the integral converges). But if you first compute the heat kernel, and then express the solution as an integral in physical space (as opposed to the Fourier domain), things converge nicely.

In Peter Miller's notes, he explains that this explicit linearization of Burgers' equation led Gardner, Green, Kruskal, and Miura to consider the Schrödinger equation as a transformation of the kdV equation. As it turns out, it was the connection to Quantum Mechanics that allowed for there to be a solution procedure for this equation.

SCATTERING AND INVERSE SCATTERING THEORY FOR THE SCHRÖDINGER EQUATION

One aspect of Quantum Mechanics is to attempt to determine properties of a "potential" by conducting experiments and measurements that are somewhat indirect. In a single dimension, this can be described quite completely. So imagine you have a "potential", which in this case is a function $V(x)$, and the laboratory is designed to perform measurements aimed at obtaining information about the solution of the Schrödinger equation (3). Basically the only dial which can be adjusted is the energy E.

We'll suppose that V decays to zero as $|x| \to \infty$. What can you measure? Bound states and reflection and transmission coefficients.