Lecture 5: The KdV equation and the Schrödinger equation

Lecture plan. We will continue the discussion of the KdV equation and the Schrödinger equation, and the amazing connection between these two equations.

The KdV equation:

$$u_t + uu_x + u_{xxx} = 0, \tag{1}$$

and the Schrödinger equation:

$$-6\psi_{xx} - u\psi = E\psi \tag{2}$$

Scattering and inverse Scattering Theory for the Schrödinger equation

One aspect of Quantum Mechanics is to attempt to determine properties of a "potential" by conducting experiments and measurements that are somewhat indirect. In a single dimension, this can be described quite completely. So imagine you have a "potential", which in this case is a function $V(x)$, and the laboratory is designed to perform measurements aimed at obtaining information about the solution of the Schrödinger equation (2). Basically the only dial which can be adjusted is the energy E.

We'll suppose that V decays to zero as $|x| \to \infty$. What can you measure? Bound states and reflection and transmission coefficients.

Now the time-dependent Schrödinger equation is

$$i\Psi_t - \Psi_{xx} + V\Psi = 0 \tag{3}$$

one arrives at the time-independent equation by making the guess $\Psi(x,t) = \psi(x)e^{iEt}$, which yields the equation

$$-E\psi - \psi_{xx} + V\psi = 0 \tag{4}$$

(modulo the constant 6, and the sign of the function u).

Assuming for a moment that the potential V is identically 0, we can solve the equation for ψ, and we find the general solution is of the form

$$\psi_0(x) = Ae^{i\sqrt{E}x} + Be^{-i\sqrt{E}x} \tag{5}$$

(assuming $E > 0$). So we find that there are solutions of the time-dependent Schrödinger equation of the form

$$\Psi(x,t) = Ae^{i\left(\sqrt{E}x + Et\right)} + Be^{-i\left(\sqrt{E}x - Et\right)}. \tag{6}$$

It is important to note: the first term represents a left-moving traveling wave, and the second term represents a right-moving traveling wave. One short-circuits this in discussion, and merely talks about the time-independent problem (for ψ) as having right- and left- moving waves, and those that behave like $e^{i\sqrt{E}x}$ are left-moving and those that behave like $e^{-i\sqrt{E}x}$ are right-moving traveling waves.

Now let us return to the case that $V \not\equiv 0$, and consider the time-independent problem for ψ. There are *fundamental normalized solutions* which are defined for E real, as follows:

$$\psi_1(x) = e^{-i\sqrt{E}x}\left(1 + \mathcal{O}\left(x^{-1}\right)\right), \text{ as } x \to +\infty, \tag{7}$$

$$\psi_2(x) = e^{i\sqrt{E}x}\left(1 + \mathcal{O}\left(x^{-1}\right)\right), \text{ as } x \to -\infty. \tag{8}$$

These solutions exist for all $\infty < x < \infty$, and possess asymptotics "at the other ∞" as follows:

$$\psi_1 = \frac{1}{T_2}e^{-i\sqrt{E}x}\left(1 + \mathcal{O}(x^{-1})\right) + \frac{R_2}{T_2}e^{i\sqrt{E}x}\left(1 + \mathcal{O}(x^{-1})\right), \quad x \to -\infty, \tag{9}$$

$$\psi_2 = \frac{1}{T_1}e^{i\sqrt{E}x}\left(1 + \mathcal{O}(x^{-1})\right) + \frac{R_1}{T_1}e^{-i\sqrt{E}x}\left(1 + \mathcal{O}(x^{-1})\right), \quad x \to -\infty\ . \tag{10}$$

By multiplying the formula (9) by T_2 (and (10) by T_1), one finds

$$T_2\psi_1 = e^{i\sqrt{E}x}\left(1+\mathcal{O}(x^{-1})\right) + R_2 e^{-i\sqrt{E}x}\left(1+\mathcal{O}(x^{-1})\right), \quad x \to -\infty, \tag{11}$$

$$T_1\psi_2 = e^{-i\sqrt{E}x}\left(1+\mathcal{O}(x^{-1})\right) + R_1 e^{i\sqrt{E}x}\left(1+\mathcal{O}(x^{-1})\right), \quad x \to -\infty\ . \tag{12}$$

We now give the interpretation of these two solutions (and their asymptotics). The solution ψ_1 represents a purely left-moving traveling wave at $+\infty$, which behaves at $-\infty$ as a sum of two traveling waves, one moving left and one moving right. So one may interpret the equation (11) for ψ_1 as follows: we have a traveling wave of unit modulus seeded at $-\infty$ which heads toward the "medium", and interacts with the potential V. There is some of the energy in this which is "transmitted" through the medium and appears as a right-moving wave at $+\infty$ (with "size" given by T_2). The remainder of the energy is "reflected" back toward $-\infty$ in the form of a left-moving traveling wave whose size is given by R_2.

There is a similar interpretation for the equation (10) which you should work out.

Here are some questions we must discuss

- Does there exist solutions as described above, and how does one prove it?
- What is the behavior of solutions for $E < 0$?
- What is the spectrum of the differential operator?
- What spectral information does one need to be able to reconstruct V?
- How does the "spectral data" evolve if we let the potential evolve according to the KdV equation?