

LECTURE 7: OUTLINE OF PROOFS RELATIVE TO SCATTERING THEORY FOR THE SCHRÖDINGER EQUATION

**Lecture plan.** We will outline the methods used to prove that there are suitably normalized solutions to the Schrödinger equation. Then we will see that the reflection coefficient evolves in a simple way as the potential  $V$  evolves according to the KdV equation.

The KdV equation:

$$(1) \quad u_t + uu_x + u_{xxx} = 0,$$

and the Schrödinger equation:

$$(2) \quad -6\psi_{xx} - V\psi = E\psi$$

INTEGRAL EQUATIONS

For this section we will consider the Schrödinger equation in the following form:

$$(3) \quad \psi'' + z^2\psi = V\psi,$$

where we have introduced a rescaling to remove the “6”,  $V$  has been replaced by  $-V$ , and  $E = z^2$ .

We will also assume that  $\psi$  is a matrix of size  $1 \times 2$ , i.e. it is a row vector. We will show that there are two normalized vector solutions to equation (3),  $\psi$  and  $\phi$ , normalized as follows:

$$(4) \quad \mathbf{u} = \psi \begin{pmatrix} e^{-izx} & 0 \\ 0 & e^{izx} \end{pmatrix} = (1, 1) + \mathcal{O}(x^{-1}), \quad \text{as } x \rightarrow +\infty$$

$$(5) \quad \mathbf{w} = \phi \begin{pmatrix} e^{-izx} & 0 \\ 0 & e^{izx} \end{pmatrix} = (1, 1) + \mathcal{O}(x^{-1}), \quad \text{as } x \rightarrow -\infty.$$

The proof requires some manipulation to show that the equation (3) and boundary condition (4) is equivalent to the following integral equation formulation:

$$(6) \quad \mathbf{u} = (1, 1) + \int_x^\infty V\mathbf{u} \begin{pmatrix} D_z(t-x) & 0 \\ 0 & D_{-z}(x-t) \end{pmatrix} dt,$$

where  $D_z(t) = \int_0^t e^{2izy} dy = \frac{1}{2iz} (e^{2izt} - 1)$ . The equation for  $\mathbf{w}$  is similar:

$$(7) \quad \mathbf{w} = (1, 1) + \int_{-\infty}^x V\mathbf{u} \begin{pmatrix} D_{-z}(x-t) & 0 \\ 0 & D_z(t-x) \end{pmatrix} dt,$$

Once you have absorbed the above calculations, then you carry out a Picard iteration, defining  $\mathbf{u}_j$  as follows:

$$(8) \quad \mathbf{u}_1 = (1, 1),$$

$$(9) \quad \mathbf{u}_{j+1} = (1, 1) + \int_x^\infty V\mathbf{u}_j \begin{pmatrix} D_z(t-x) & 0 \\ 0 & D_{-z}(x-t) \end{pmatrix} dt,$$

and then working hard to prove that this sequence converges. You basically have to estimate  $\mathbf{u}_{j+1} - \mathbf{u}_j$ , which boils down to proving that the integral operator

$$(10) \quad \mathcal{I}(\mathbf{h}) = \int_x^\infty V\mathbf{h}_j \begin{pmatrix} D_z(t-x) & 0 \\ 0 & D_{-z}(x-t) \end{pmatrix} dt$$

is a contraction.

WHAT HAPPENS IF  $V = V(x, t)$ ?

Now let's consider (finally!) the interesting situation that  $V$  depends parametrically on the parameter  $t$ . The first thing to do is consider not one but two operators. The first is our old friend,

$$(11) \quad L = -\frac{d^2}{dx^2} + V,$$

and the second is a more foul beast:

$$(12) \quad B = -4\frac{d^3}{dx^3} + 3\frac{d}{dx} V + 3V\frac{d}{dx}.$$

You can verify by algebra and calculus that, as an operator acting on functions of  $x$ , we have

$$(13) \quad B = 4\frac{d}{dx}L - \frac{d}{dx} V + 3V\frac{d}{dx}.$$

Now there is some magic, which is the following. Let us suppose that the potential  $V$  depends parametrically on  $t$ , and then differentiate these operators with respect to  $t$ . Here is an amazing equation:

$$(14) \quad \frac{\partial}{\partial t}L - (BL - LB) = (V_t - 6VV_x + V_{xxx}) \cdot \cdot$$

So somehow KdV is equivalent to the operator equation

$$(15) \quad L_t = BL - LB$$

which is referred to as a Lax equation for the KdV equation. The pair of operators  $L$  and  $B$  are referred to as a Lax-pair for the KdV equation.

There are many different ways to formulate this amazing fact, which will be discussed in class. For now, let's see if we can play around with the normalized solution  $\psi$  a little, and seek a function  $\hat{\psi}$  which simultaneously solves the following two equations:

$$(16) \quad L\hat{\psi} = z^2\hat{\psi},$$

$$(17) \quad \hat{\psi}_t = B\hat{\psi}.$$

First observe that  $\psi_t - B\psi$  also solves the operator equation

$$(18) \quad L(\psi_t - B\psi) = z^2(\psi_t - B\psi)$$

and thus must be related to the (fundamental) solution  $\psi$ , i.e. one must have

$$(19) \quad \psi_t - B\psi = \psi \text{Const}$$

To find Const, let  $x \rightarrow \infty$ , and assuming (justifiably) uniform behavior in this limit, one finds

$$(20) \quad \text{Const} = -4iz^3\sigma_3 = -4iz^3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then, setting  $\hat{\psi} = \psi C(t)$ , and assuming that  $\hat{\psi}$  is our desired simultaneous solution, you may verify that  $C(t)$  can be found in closed form:

$$(21) \quad C(t) = e^{4iz^3t\sigma_3} C_0 = \begin{pmatrix} e^{4iz^3t} & 0 \\ 0 & e^{-4iz^3t} \end{pmatrix}.$$