Lecture 8: Time evolution of the scattering data under the KdV equation.

Lecture plan. We’ll continue our discussion of the evolution of the scattering data as the potential $V$ evolves according to the KdV equation.

The KdV equation:

\[ u_t + uu_x + u_{xxx} = 0, \]

and the Schrödinger equation:

\[ \text{What happens if } V = V(x,t)? \]

Now let’s consider (finally!) the interesting situation that $V$ depends parametrically on the parameter $t$. The first thing to do is consider not one but two operators. The first is our old friend,

\[ L = -\frac{d^2}{dx^2} + V, \]

and the second is a more foul beast:

\[ B = -4\frac{d^3}{dx^3} + 3\frac{d}{dx} V + 3V\frac{d}{dx}. \]

You can verify by algebra and calculus that, as an operator acting on functions of $x$, we have

\[ B = 4\frac{d}{dx}L - \frac{d}{dx} V + 3V\frac{d}{dx}. \]

Now there is some magic, which is the following. Let us suppose that the potential $V$ depends parametrically on $t$, and then differentiate these operators with respect to $t$. Here is an amazing equation:

\[ \frac{\partial}{\partial t} L - (B L - L B) = (V_t - 6VV_x + V_{xxx}) \cdot . \]

So somehow KdV is equivalent to the operator equation

\[ L_t = BL - LB \]

which is referred to as a Lax equation for the KdV equation. The pair of operators $L$ and $B$ are referred to as a Lax-pair for the KdV equation.

In the previous Lecture (Lecture 07) we showed that there is a simultaneous solution $\hat{\psi}$ to the pair of equations

\[ \hat{L}\hat{\psi} = z^2 \hat{\psi}, \]

\[ \hat{\psi}_t = B\hat{\psi}. \]

The solution is of the form \( \hat{\psi} = \psi C(t) \), where

\[ C(t) = e^{4iz^2t} C_0 = \begin{pmatrix} e^{4iz^2t} & 0 \\ 0 & e^{-4iz^2t} \end{pmatrix}. \]

You can similarly verify that $\hat{\phi} = \phi C(t)$ is also a simultaneous solution of the pair of equations (7).

EvoLution of the Scattering Data

Recall that we have these two normalized solutions $\psi$ and $\phi$ to the Schrödinger equation. They are related, as we have discussed before. We will express this relationship using the scattering matrix, as follows:

\[ \psi(x,z) = \phi(x,z)S(z,t). \]
By comparing to formulae (6) and (6) of Lecture 06 with formulae (4) and (5) of Lecture 07, one may verify that

\begin{align}
S_{11}(z,t) &= \frac{R_2}{T_2}, \\
S_{21}(z,t) &= \frac{1}{T_2}, \\
S_{12}(z,t) &= \frac{1}{T_1}, \\
S_{22}(z,t) &= \frac{R_1}{T_1}.
\end{align}

Now since \( \hat{\psi} \) and \( \hat{\phi} \) are both simultaneous solutions of (7) we have immediately that

\begin{align}
\hat{S} := C(t)^{-1} S(z,t) C(t)
\end{align}

is independent of \( t \)!

So if we can determine the scattering matrix for the initial data, we then know the scattering data for the potential at all later times, \( t > 0 \)!

What about the eigenvalues and eigenfunctions?

Let us suppose that we have an eigenvalue \( E_j(t) < 0 \), and the eigenfunction is \( f_j(x,t) \), assumed to be normalized so that \( ||f_j||_{L^2} = 1 \). One may again verify that \( f_j(x,t) \) satisfies the following:

\begin{align}
L(\partial_t f_j - B f_j) &= E_j \left( \partial_t f_j - B f_j \right) + (\partial_t E_j) f_j.
\end{align}

Now \( L \) is self-adjoint with respect to the standard inner-product on \( L^2 \):

\begin{align}
\int_{\mathbb{R}} a(x) b(x) dx &= \int_{\mathbb{R}} L a(x) b(x) dx,
\end{align}

and so, multiplying (20) by \( f_j \) and integrating over \( \mathbb{R} \), we find

\begin{align}
0 &= \partial_t E_j,
\end{align}

so the eigenvalues are independent of time!

Now we see that \( \partial_t f_j - B f_j \) is again an eigenfunction, and so

\begin{align}
\partial_t f_j - B f_j &= c_j f_j.
\end{align}

Again multiplying by \( \overline{f_j} \) and integrating, (this time using the fact that \( B \) is skew-adjoint, i.e. \( \int_{\mathbb{R}} a(x) b(x) dx = - \int_{\mathbb{R}} B a(x) b(x) dx \)) you can check that \( c_j(t) = 0 \), and so

\begin{align}
\partial_t f_j &= B f_j.
\end{align}

Now recall the behavior for \( x \to \pm \infty \):

\begin{align}
f_j &= b_{\pm}(t) e^{-\sqrt{-E_j} |x|} \left( 1 + O \left( x^{-1} \right) \right).
\end{align}

We can deduce the evolution of the constants \( b_{\pm}(t) \) by differentiating this relationship with respect to \( t \), which yields

\begin{align}
\partial_t b_{\pm}(t) e^{-\sqrt{-E_j} |x|} \left( 1 + O \left( x^{-1} \right) \right) &= B b_{\pm}(t) e^{-\sqrt{-E_j} |x|} \left( 1 + O \left( x^{-1} \right) \right) \\
&= \mp 4 \left( -\sqrt{-E_j} \right)^3 b_{\pm}(t) e^{-\sqrt{-E_j} |x|} \left( 1 + O \left( x^{-1} \right) \right)
\end{align}
and so

(26) \[ b_{\pm}(t) = b_{\pm}(0)e^{\pm 4t(-E_j)^{3/2}} \]

Thus we have established that the full scattering data associated to the potential $V$, once determined at $t = 0$, is known for all positive times $t > 0$, if $V$ evolves according to the KdV equation!

What can we say about the solution?