Lecture 9: Inverse scattering theory overview, a few research projects, and other examples

Lecture plan. We will summarize inverse scattering theory, which yields the potential V if one is given the scattering data, as described in previous Lectures. Then we will discuss some research projects, and then we will consider other examples of nonlinear pdes or odes that are solvable by scattering / inverse scattering theory techniques.

Inverse scattering theory

Recall from Lecture 7 that we have shown how one can prove that there are two normalized vector solutions to the Schrödinger equation, ψ and ϕ, normalized as follows:

$$\mathbf{u} = \psi \begin{pmatrix} e^{-izx} & 0 \\ 0 & e^{izx} \end{pmatrix} = (1,1) + \mathcal{O}\left(x^{-1}\right), \quad \text{as } x \to +\infty \tag{1}$$

$$\mathbf{w} = \phi \begin{pmatrix} e^{-izx} & 0 \\ 0 & e^{izx} \end{pmatrix} = (1,1) + \mathcal{O}\left(x^{-1}\right), \quad \text{as } x \to -\infty. \tag{2}$$

The fundamental observation is that these odes are equivalent to integral equations:

$$\mathbf{u} = (1,1) + \int_x^{\infty} V\mathbf{u} \begin{pmatrix} D_z(t-x) & 0 \\ 0 & D_{-z}(t-x) \end{pmatrix} dt, \tag{3}$$

where $D_z(t) = \int_0^t e^{2izy}dy = \frac{1}{2iz}\left(e^{2izt}-1\right)$. The equation for $\mathbf{w}$ is similar:

$$\mathbf{w} = (1,1) + \int_{-\infty}^{x} V\mathbf{u} \begin{pmatrix} D_{-z}(x-t) & 0 \\ 0 & D_z(x-t) \end{pmatrix} dt\ . \tag{4}$$

Note that I have corrected a typo which appeared in Lecture 7: the $(2,2)$ in the matrices appearing above were incorrect. It would be good to double check that these are now correct! and then we constructed a standard iteration scheme:

$$\mathbf{u}_1 = (1,1), \tag{5}$$

$$\mathbf{u}_{j+1} = (1,1) + \int_x^{\infty} V\mathbf{u}_j \begin{pmatrix} D_z(t-x) & 0 \\ 0 & D_{-z}(t-x) \end{pmatrix} dt, \tag{6}$$

which is possible to prove converges.

Inverse scattering theory is, in a formal sense, seeking a list of properties satisfied by the functions ψ, ϕ, and also the eigenvalues and eigenfunctions, which are *perfect* in the sense that there is a 1-1 correspondence between the *potential* and the *list of properties.*

Note that the quantity $D_z(x)$ can be extended analytically to the complex z plane. This indicates that $\mathbf{u}$ and $\mathbf{w}$ might be analytic functions of z in some sense. Of course things are more subtle than this: for the integrals defining $\mathbf{u}$ and $\mathbf{w}$ are over semi-infinite integrals. So, for example, if we are to define $\mathbf{u}$ for $z \in \mathbb{C}_+$, we would require that $D_z(t-x)$ and $D_{-z}(t-x)$ be decaying as $t \to +\infty$. However, only one works! What this means is:

- $\mathbf{u}_1$ can be extended analytically to $\mathbb{C}_+$, and $\mathbf{u}_2$ can be analytically extended to $\mathbb{C}_-$.
- $\mathbf{w}_2$ can be extended analytically to $\mathbb{C}_+$, and $\mathbf{w}_1$ can be analytically extended to $\mathbb{C}_-$.

Now define $\mathbf{M}(z)$ as follows:

$$\mathbf{M} = \left(\mathbf{u}_1,\ \frac{1}{\mathbf{S}_{11}(z)}\mathbf{w}_2\right) \quad \text{for } z \in \mathbb{C}_+ \tag{7}$$

$$\mathbf{M} = \left(\frac{1}{\mathbf{S}_{11}(z)}\mathbf{w}_1,\ \mathbf{u}_2\right) \quad \text{for } z \in \mathbb{C}_-. \tag{8}$$

The point *should be*, if things are defined correctly, that $\mathbf{M}$ converges to $(1,1)$ as $x \to +\infty$ for $z \in \mathbb{C}_+$, and similarly $\mathbf{M}$ converges to $(1,1)$ as $x \to -\infty$ for $z \in \mathbb{C}_-$. During Lecture 9 we didn't manage to see this, so we'll return to it in Lecture 10.