LECTURE 12: RANDOM MATRICES AND ORTHOGONAL POLYNOMIALS

Lecture plan. We will first continue with our overview of random matrix theory for 5-10 minutes, then start establishing the connection to orthogonal polynomials.

THE CONNECTION TO ORTHOGONAL POLYNOMIALS

We have the probability distribution function on *ordered eigenvalues*,

$$\mathbb{P}(\lambda_1,\dots,\lambda_N) = \frac{1}{Z_N} e^{-\frac{1}{2}\sum_{j=1}^N \lambda_j^2} \prod_{1\le j<k\le N} (\lambda_k-\lambda_j)^2\,, \tag{1}$$

and we have defined the orthogonal polynomials $p_j(x) = \kappa_j x^j + \text{l.o.t.}$, $\kappa_j > 0$, via

$$\int_{\mathbb{R}} p_j(x) p_k(x) e^{-\frac{1}{2}x^2}\,dx = \delta_{jk}. \tag{2}$$

Closely related to these orthogonal polynomials is the *reproducing kernel*,

$$K_N(x,y) = e^{-\frac{1}{4}\left(x^2+y^2\right)} \sum_{\ell=0}^{N-1} p_\ell(x) p_\ell(y). \tag{3}$$

We want to establish (first) the following formula:

$$\mathbb{P}(\lambda_1,\dots,\lambda_N) = \det\begin{pmatrix} K_N(\lambda_1,\lambda_1) & \cdots & K_N(\lambda_1,\lambda_N) \\ \vdots & \ddots & \vdots \\ K_N(\lambda_N,\lambda_1) & \cdots & K_N(\lambda_N,\lambda_N)\end{pmatrix}. \tag{4}$$

We establish (4) by starting with the product appearing in (1) (without the squares), and using some basic linear algebra:

$$\prod_{1\le j<k\le N} (\lambda_k-\lambda_j) = \det\begin{pmatrix} 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_N \\ \vdots & \ddots & \vdots \\ \lambda_1^{N-1} & \cdots & \lambda_N^{N-1}\end{pmatrix}. \tag{5}$$

Now, by multi-linearity of the determinant, I can replace the second row in (6) by itself, plus any multiple of the first row. In other words, if $\pi_1(x)$ is any monic first degree polynomial, then we can replace the second row by $(\pi_1(\lambda_1),\dots,\pi_1(\lambda_N))$

Similarly we may use any monic second degree polynomial $\pi_2(x)$ and replace the third row by $(\pi_2(\lambda_1),\dots,\pi_2(\lambda_N))$, and we may continue in this way through to the last row.

Thus

$$\prod_{1\le j<k\le N} (\lambda_k-\lambda_j) = \det\begin{pmatrix} 1 & \cdots & 1 \\ \pi_1(\lambda_1) & \cdots & \pi_1(\lambda_N) \\ \vdots & \ddots & \vdots \\ \pi_{N-1}(\lambda_1) & \cdots & \pi_{N-1}(\lambda_N)\end{pmatrix}. \tag{6}$$

Now, it is straightforward to see that if $\kappa_j > 0$ for each j, then we have

$$\prod_{1\le j<k\le N} (\lambda_k-\lambda_j) = \frac{1}{\prod_{j=0}^{N-1}\kappa_j}\det\begin{pmatrix} \kappa_0 & \cdots & \kappa_0 \\ \kappa_1\pi_1(\lambda_1) & \cdots & \kappa_1\pi_1(\lambda_N) \\ \vdots & \ddots & \vdots \\ \kappa_{N-1}\pi_{N-1}(\lambda_1) & \cdots & \kappa_{N-1}\pi_{N-1}(\lambda_N)\end{pmatrix}. \tag{7}$$

Note that now we are at liberty to use *any polynomial of degree j that we want* in the $j+1$st column! (So long as the leading coefficient is nonzero.)

One of our acceptable choices is to use, in each row, the appropriate orthogonal polynomial. Doing this, we have

$$\prod_{1\le j<k\le N} (\lambda_k - \lambda_j) = \frac{1}{\prod_{j=0}^{N-1} \kappa_j} \det \begin{pmatrix} p_0(\lambda_1) & \cdots & p_0(\lambda_N) \\ p_1(\lambda_1) & \cdots & p_1(\lambda_N) \\ \vdots & \ddots & \vdots \\ p_{N-1}(\lambda_1) & \cdots & p_{N-1}(\lambda_N) \end{pmatrix}. \tag{8}$$

Now, since $\det M = \det M^T$, we have

$$\prod_{1\le j<k\le N} (\lambda_k - \lambda_j)^2 = \frac{1}{\prod_{j=0}^{N-1} \kappa_j^2} \det \begin{pmatrix} p_0(\lambda_1) & \cdots & p_0(\lambda_N) \\ p_1(\lambda_1) & \cdots & p_1(\lambda_N) \\ \vdots & \ddots & \vdots \\ p_{N-1}(\lambda_1) & \cdots & p_{N-1}(\lambda_N) \end{pmatrix}^T \cdot \tag{9}$$

$$\cdot \det \begin{pmatrix} p_0(\lambda_1) & \cdots & p_0(\lambda_N) \\ p_1(\lambda_1) & \cdots & p_1(\lambda_N) \\ \vdots & \ddots & \vdots \\ p_{N-1}(\lambda_1) & \cdots & p_{N-1}(\lambda_N) \end{pmatrix}$$

And so our probability distribution function (1) can be written in the form

$$\mathbb{P}(\lambda_1,\dots,\lambda_N) = \frac{1}{Z_N \prod_{j=0}^{N-1} \kappa_j^2} \det \begin{pmatrix} e^{-\frac14 \lambda_1^2} p_0(\lambda_1) & \cdots & e^{-\frac14 \lambda_N^2} p_0(\lambda_N) \\ e^{-\frac14 \lambda_1^2} p_1(\lambda_1) & \cdots & e^{-\frac14 \lambda_N^2} p_1(\lambda_N) \\ \vdots & \ddots & \vdots \\ e^{-\frac14 \lambda_1^2} p_{N-1}(\lambda_1) & \cdots & e^{-\frac14 \lambda_N^2} p_{N-1}(\lambda_N) \end{pmatrix}^T \cdot$$

$$\cdot \det \begin{pmatrix} e^{-\frac14 \lambda_1^2} p_0(\lambda_1) & \cdots & e^{-\frac14 \lambda_N^2} p_0(\lambda_N) \\ e^{-\frac14 \lambda_1^2} p_1(\lambda_1) & \cdots & e^{-\frac14 \lambda_N^2} p_1(\lambda_N) \\ \vdots & \ddots & \vdots \\ e^{-\frac14 \lambda_1^2} p_{N-1}(\lambda_1) & \cdots & e^{-\frac14 \lambda_N^2} p_{N-1}(\lambda_N) \end{pmatrix}$$

$$= \frac{1}{Z_N \prod_{j=0}^{N-1} \kappa_j^2} \times$$

$$\det \begin{pmatrix} e^{-\frac14 \left(\lambda_1^2+\lambda_1^2\right)} \sum_{\ell=0}^{N-1} p_\ell(\lambda_1) p_\ell(\lambda_1) & \cdots & e^{-\frac14 \left(\lambda_1^2+\lambda_{N-1}^2\right)} \sum_{\ell=0}^{N-1} p_\ell(\lambda_1) p_\ell(\lambda_{N-1}) \\ e^{-\frac14 \left(\lambda_2^2+\lambda_1^2\right)} \sum_{\ell=0}^{N-1} p_\ell(\lambda_2) p_\ell(\lambda_1) & \cdots & e^{-\frac14 \left(\lambda_2^2+\lambda_{N-1}^2\right)} \sum_{\ell=0}^{N-1} p_\ell(\lambda_2) p_\ell(\lambda_{N-1}) \\ \vdots & \ddots & \vdots \\ e^{-\frac14 \left(\lambda_{N-1}^2+\lambda_1^2\right)} \sum_{\ell=0}^{N-1} p_\ell(\lambda_{N-1}) p_\ell(\lambda_1) & \cdots & e^{-\frac14 \left(\lambda_{N-1}^2+\lambda_{N-1}^2\right)} \sum_{\ell=0}^{N-1} p_\ell(\lambda_{N-1}) p_\ell(\lambda_{N-1}) \end{pmatrix},$$

as desired. Now rather than considering a probability distribution function defined on the set of all *ordered eigenvalues*, it is simpler to e working with the set $\mathbb{R}^N$. To accomplish this, we will consider functions of the eigenvalues that are *symmetric*, i.e. we will be considering integrals (with respect to the probability distribution function $\mathbb{P}$) of functions f that satisfy

$$f(\lambda_1,\dots,\lambda_N) = f\left(\lambda_{\sigma(1)},\dots,\lambda_{\sigma(N)}\right), \tag{10}$$

for all permutations σ. For such functions, we have

$$\int_{\lambda_1\le\cdots\le\lambda_N} f(\lambda_1,\dots,\lambda_N)\, \mathbb{P}(\lambda_1,\dots,\lambda_N)\, d^N\lambda = \frac{1}{N!} \sum_\sigma \int_{\lambda_{\sigma(1)}\le\cdots\le\lambda_{\sigma(N)}} f\left(\lambda_{\sigma(1)},\dots,\lambda_{\sigma(N)}\right) \mathbb{P}\left(\lambda_{\sigma(1)},\dots,\lambda_{\sigma(N)}\right) d^N\lambda$$

$$= \frac{1}{N!} \sum_\sigma \int_{\lambda_{\sigma(1)}\le\cdots\le\lambda_{\sigma(N)}} f(\lambda_1,\dots,\lambda_N)\, \mathbb{P}(\lambda_1,\dots,\lambda_N)\, d^N\lambda$$

$$= \frac{1}{N!} \int_{\mathbb{R}^N} f(\lambda_1,\dots,\lambda_N)\, \mathbb{P}(\lambda_1,\dots,\lambda_N)\, d^N\lambda.$$

The next important trick is to see what happens if we integrate one of the variables. Because the probability distribution function is symmetric, we will compute

$$\int_{\mathbb{R}} \mathbb{P}\left(\lambda_1, \ldots, \lambda_N\right)\ d\lambda_N = \int_{\mathbb{R}} \frac{1}{Z_N \prod_{j=0}^{N-1} \kappa_j^2} \det \begin{pmatrix} K_N(\lambda_1,\lambda_1) & \cdots & K_N(\lambda_1,\lambda_N) \\ \vdots & \ddots & \vdots \\ K_N(\lambda_N,\lambda_1) & \cdots & K_N(\lambda_N,\lambda_N) \end{pmatrix} d\lambda_N \tag{11}$$

$$= \frac{1}{Z_N \prod_{j=0}^{N-1} \kappa_j^2} \int_{\mathbb{R}} \left(\sum_{j=1}^{N} (-1)^{N+j} K_N\left(\lambda_N, \lambda_j\right) \det \begin{pmatrix} K_N(\lambda_1,\lambda_1) & \cdots & K_N(\lambda_1,\lambda_N) \\ \vdots & \ddots & \vdots \\ K_N(\lambda_N,\lambda_1) & \cdots & K_N(\lambda_N,\lambda_N) \end{pmatrix}^{(N,j)} \right) d\lambda_N \tag{12}$$

where the notation $A^{(k,j)}$ means the matrix formed by starting with A, and excising the kth row, and the jth column.

Next, we expand each subsequent determinant (except the last one) along the last column:

$$\int_{\mathbb{R}} \mathbb{P}\left(\lambda_1, \ldots, \lambda_N\right)\ d\lambda_N = \frac{1}{Z_N \prod_{j=0}^{N-1} \kappa_j^2} \int_{\mathbb{R}}$$
$$\left(\sum_{j=1}^{N-1} \sum_{k=1}^{N-1} (-1)^{j+k-1} K_N\left(\lambda_N, \lambda_j\right) K_N\left(\lambda_k, \lambda_N\right) \det \begin{pmatrix} K_N(\lambda_1,\lambda_1) & \cdots & K_N(\lambda_1,\lambda_N) \\ \vdots & \ddots & \vdots \\ K_N(\lambda_N,\lambda_1) & \cdots & K_N(\lambda_N,\lambda_N) \end{pmatrix}^{(N,j)(k,N)} \right) d\lambda_N$$
$$+ \frac{1}{Z_N \prod_{j=0}^{N-1} \kappa_j^2} \int_{\mathbb{R}} K_N\left(\lambda_N, \lambda_N\right) \det \begin{pmatrix} K_N(\lambda_1,\lambda_1) & \cdots & K_N(\lambda_1,\lambda_N) \\ \vdots & \ddots & \vdots \\ K_N(\lambda_N,\lambda_1) & \cdots & K_N(\lambda_N,\lambda_N) \end{pmatrix}^{(N,N)} d\lambda_N\ .$$

Now the notation is quite cumbersome, but the important thing to note is that the determinants are all *independent of* λ_N*!* So they can be factored out of the integrals. What we are left staring at is

$$\int_{\mathbb{R}} \mathbb{P}\left(\lambda_1, \ldots, \lambda_N\right)\ d\lambda_N = \frac{1}{Z_N \prod_{j=0}^{N-1} \kappa_j^2} \sum_{j=1}^{N-1} \sum_{k=1}^{N-1} (-1)^{j+k-1} \det \begin{pmatrix} K_N(\lambda_1,\lambda_1) & \cdots & K_N(\lambda_1,\lambda_N) \\ \vdots & \ddots & \vdots \\ K_N(\lambda_N,\lambda_1) & \cdots & K_N(\lambda_N,\lambda_N) \end{pmatrix}^{(N,j)(k,N)} \times$$
$$\times \int_{\mathbb{R}} K_N\left(\lambda_N, \lambda_j\right) K_N\left(\lambda_k, \lambda_N\right) d\lambda_N$$
$$+ \frac{1}{Z_N \prod_{j=0}^{N-1} \kappa_j^2} \det \begin{pmatrix} K_N(\lambda_1,\lambda_1) & \cdots & K_N(\lambda_1,\lambda_N) \\ \vdots & \ddots & \vdots \\ K_N(\lambda_N,\lambda_1) & \cdots & K_N(\lambda_N,\lambda_N) \end{pmatrix}^{(N,N)} \int_{\mathbb{R}} K_N\left(\lambda_N, \lambda_N\right)\ d\lambda_N\ .$$

Finally, the two integrals above can be evaluated using the orthogonality relations:

$$\int_{\mathbb{R}} K_N(\lambda_N,\lambda_N) d\lambda_N = N, \qquad \int_{\mathbb{R}} K_N(\lambda_N,\lambda_j) K_N(\lambda_k,\lambda_N) d\lambda_N = K_N(\lambda_k,\lambda_j), \tag{13}$$

and so we have

$$\int_{\mathbb{R}} \mathbb{P}\left(\lambda_1, \ldots, \lambda_N\right)\ d\lambda_N = \frac{1}{Z_N \prod_{j=0}^{N-1} \kappa_j^2} \sum_{j=1}^{N-1} \sum_{k=1}^{N-1} (-1)^{j+k-1} \det \begin{pmatrix} K_N(\lambda_1,\lambda_1) & \cdots & K_N(\lambda_1,\lambda_N) \\ \vdots & \ddots & \vdots \\ K_N(\lambda_N,\lambda_1) & \cdots & K_N(\lambda_N,\lambda_N) \end{pmatrix}^{(N,j)(k,N)} \times$$
$$\times K_N(\lambda_k,\lambda_j)$$
$$+ \frac{N}{Z_N \prod_{j=0}^{N-1} \kappa_j^2} \det \begin{pmatrix} K_N(\lambda_1,\lambda_1) & \cdots & K_N(\lambda_1,\lambda_N) \\ \vdots & \ddots & \vdots \\ K_N(\lambda_N,\lambda_1) & \cdots & K_N(\lambda_N,\lambda_N) \end{pmatrix}^{(N,N)} .$$

Now the inner summation can be evaluated as a determinant:

$$\sum_{k=1}^{N-1}(-1)^{j+k-1}\det\begin{pmatrix} K_N(\lambda_1,\lambda_1) & \cdots & K_N(\lambda_1,\lambda_N)\\ \vdots & \ddots & \vdots\\ K_N(\lambda_N,\lambda_1) & \cdots & K_N(\lambda_N,\lambda_N)\end{pmatrix}^{(N,j)(k,N)} \times K_N(\lambda_k,\lambda_j) =$$

$$= -\det\begin{pmatrix} K_N(\lambda_1,\lambda_1) & \cdots & K_N(\lambda_1,\lambda_N)\\ \vdots & \ddots & \vdots\\ K_N(\lambda_N,\lambda_1) & \cdots & K_N(\lambda_N,\lambda_N)\end{pmatrix}^{(N,N)} = -\det\begin{pmatrix} K_N(\lambda_1,\lambda_1) & \cdots & K_N(\lambda_1,\lambda_{N-1})\\ \vdots & \ddots & \vdots\\ K_N(\lambda_{N-1},\lambda_1) & \cdots & K_N(\lambda_{N-1},\lambda_{N-1})\end{pmatrix}$$

And so we finally have

$$\int_{\mathbb{R}}\mathbb{P}\left(\lambda_1,\dots,\lambda_N\right)\,d\lambda_N = \frac{1}{Z_N\prod_{j=0}^{N-1}\kappa_j^2}\sum_{j=1}^{N-1}\left(-\det\begin{pmatrix} K_N(\lambda_1,\lambda_1) & \cdots & K_N(\lambda_1,\lambda_{N-1})\\ \vdots & \ddots & \vdots\\ K_N(\lambda_{N-1},\lambda_1) & \cdots & K_N(\lambda_{N-1},\lambda_{N-1})\end{pmatrix}\right)$$

$$+\frac{N}{Z_N\prod_{j=0}^{N-1}\kappa_j^2}\det\begin{pmatrix} K_N(\lambda_1,\lambda_1) & \cdots & K_N(\lambda_1,\lambda_{N-1})\\ \vdots & \ddots & \vdots\\ K_N(\lambda_{N-1},\lambda_1) & \cdots & K_N(\lambda_{N-1},\lambda_{N-1})\end{pmatrix}.$$

This becomes, upon simplifying,

$$\int_{\mathbb{R}}\mathbb{P}\left(\lambda_1,\dots,\lambda_N\right)\,d\lambda_N = \frac{1}{Z_N\prod_{j=0}^{N-1}\kappa_j^2}\det\begin{pmatrix} K_N(\lambda_1,\lambda_1) & \cdots & K_N(\lambda_1,\lambda_{N-1})\\ \vdots & \ddots & \vdots\\ K_N(\lambda_{N-1},\lambda_1) & \cdots & K_N(\lambda_{N-1},\lambda_{N-1})\end{pmatrix}. \tag{14}$$

If we subsequently evaluate the integral of *this quantity* with respect to λ_{N-1}, we find

$$\iint_{\mathbb{R}^2}\mathbb{P}\left(\lambda_1,\dots,\lambda_N\right)\,d\lambda_N d\lambda_{N-1} = \frac{2}{Z_N\prod_{j=0}^{N-1}\kappa_j^2}\det\begin{pmatrix} K_N(\lambda_1,\lambda_1) & \cdots & K_N(\lambda_1,\lambda_{N-2})\\ \vdots & \ddots & \vdots\\ K_N(\lambda_{N-2},\lambda_1) & \cdots & K_N(\lambda_{N-2},\lambda_{N-2})\end{pmatrix}. \tag{15}$$

(Can you determine from whence came the factor of 2?)

Proceeding further, we have

$$\int_{\mathbb{R}^j}\mathbb{P}\left(\lambda_1,\dots,\lambda_N\right)\,d\lambda_N\cdots d\lambda_{N-(j+1)} = \frac{j!}{Z_N\prod_{j=0}^{N-1}\kappa_j^2}\det\begin{pmatrix} K_N(\lambda_1,\lambda_1) & \cdots & K_N(\lambda_1,\lambda_{N-j})\\ \vdots & \ddots & \vdots\\ K_N(\lambda_{N-j},\lambda_1) & \cdots & K_N(\lambda_{N-j},\lambda_{N-j})\end{pmatrix}.$$

Thus

$$\int_{\mathbb{R}^{N-1}}\mathbb{P}\left(\lambda_1,\dots,\lambda_N\right)\,d\lambda_N\cdots d\lambda_2 = \frac{(N-1)!}{Z_N\prod_{j=0}^{N-1}\kappa_j^2}K_N(\lambda_1,\lambda_1)\,,$$

and, evaluating this last integral, we find (since $\frac{1}{N!}\mathbb{P}$ is a probability measure on $\mathbb{R}^N$) that

$$1 = \frac{1}{N!}\int_{\mathbb{R}^N}\mathbb{P}\left(\lambda_1,\dots,\lambda_N\right)\,d\lambda_N\cdots d\lambda_1 = \frac{1}{Z_N\prod_{j=0}^{N-1}\kappa_j^2}\,,$$

and we finally have established (4), which we re-write here for emphasis:

$$\mathbb{P}(\lambda_1,\dots,\lambda_N) = \det\begin{pmatrix} K_N(\lambda_1,\lambda_1) & \cdots & K_N(\lambda_1,\lambda_N)\\ \vdots & \ddots & \vdots\\ K_N(\lambda_N,\lambda_1) & \cdots & K_N(\lambda_N,\lambda_N)\end{pmatrix}. \tag{16}$$