Lecture 13: Mean Density of Eigenvalues

Lecture plan. We will first continue with the connection to orthogonal polynomials. Then we will discuss in great detail the mean density of eigenvalues.

1. *Generality with regard to the “external field”* $V$

A brief observation: If we consider the probability measure on matrices defined by

$$ \frac{1}{Z_N} e^{-\text{Tr} V(M)} dM $$

with $V(x)$ taken to be a polynomial (or more general, scalar-valued function) of the scalar variable $x$, then the induced measure on eigenvalues is

$$ p(\lambda_1, \ldots, \lambda_N) = \frac{1}{Z_N} e^{-\sum_{j=1}^N V(\lambda_j)} \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2, $$

and the calculations of Lecture 12 may be applied directly, but rather than using Hermite polynomials, one uses instead the polynomials that are orthogonal with respect to the measure $e^{-V(x)} dx$.

The mean density of eigenvalues

In this Lecture we will find a formula for the mean density of eigenvalues. We first define this quantity, as follows. First, consider the random variable which is the fraction of eigenvalues less than $x$:

$$ \frac{1}{N} \# \{ \text{eigenvalues } \lambda_k \text{ s.t. } \lambda_k < x \}. $$

It will prove useful to define the characteristic function of the half-line $(-\infty, x)$:

$$ \Xi_x(\lambda) = \begin{cases} 1 & \text{if } \lambda < x \\ 0 & \text{if } \lambda \geq x \end{cases}. $$

Clearly, if one selects a matrix $M$ at random according to our probability distribution function, and then computes its eigenvalues $\lambda_1 \leq \ldots \leq \lambda_N$, then

$$ \frac{1}{N} \# \{ \text{eigenvalues } \lambda_k \text{ s.t. } \lambda_k < x \} = \frac{1}{N} \sum_{j=1}^N \Xi_x(\lambda_j), $$

We compute the average of (5) with respect to the probability distribution function, and we have the mean fraction of eigenvalues less than $x$:

$$ \mathbb{E} \left( \frac{1}{N} \# \{ \text{eigenvalues } \lambda_k \text{ s.t. } \lambda_k < x \} \right) = \frac{1}{N!} \int_{\mathbb{R}^N} p(\lambda_1, \ldots, \lambda_N) \sum_{j=1}^N \Xi_{(-\infty, x)}(\lambda_j) \, d^N \lambda $$

$$ = \frac{1}{N!} \sum_{j=1}^N \int_{\mathbb{R}^N} p(\lambda_1, \ldots, \lambda_N) \Xi_{(-\infty, x)}(\lambda_j) \, d^N \lambda $$

(note that the function $\sum_{j=1}^N \Xi_x(\lambda_j)$ is a symmetric function, and so we may integrate over $\mathbb{R}^N$ instead of over the ordered eigenvalues). We use this symmetry again, to introduce a change of variables $\tilde{\lambda} \mapsto \tilde{\lambda}'$ (which we again call $\tilde{\lambda}$) so that for each summand, $\lambda_j \mapsto \lambda_j' (= \lambda_1)$.

$$ \mathbb{E} \left( \frac{1}{N} \# \{ \text{eigenvalues } \lambda_j \text{ s.t. } \lambda_j < x \} \right) = \frac{1}{N!} \sum_{j=1}^N \int_{\mathbb{R}^N} p(\lambda_1, \ldots, \lambda_N) \Xi_{(-\infty, x)}(\lambda_1) \, d^N \lambda $$

$$ = \frac{1}{N!} \int_{\mathbb{R}^N} p(\lambda_1, \ldots, \lambda_N) \Xi_{(-\infty, x)}(\lambda_1) \, d^N \lambda $$

Using the tricks of integration from Lecture 12, we may evaluate this by viewing it as an iterated integral:
From this formula we learn many things. First, the mean fraction of eigenvalues is an analytic function of $x$. Second, the mean density of eigenvalues, $\rho_1^{(N)}(x)$, defined to be the derivative of the mean fraction of eigenvalues, is given by

\[
\rho_1^{(N)}(x) = \frac{1}{N} K_N(x, x) = e^{-\frac{1}{2}x^2} \sum_{\ell=0}^{N-1} p_\ell(x)^2 .
\]

Now let’s take a look at some plots of this function as $N$ grows. Remember, our overarching goal is to understand the statistical behavior of these matrices for large $N$!