Lecture 15: Eigenvalue statistics, and the connection between orthogonal polynomials and Riemann–Hilbert problems.

Lecture plan. We will discuss more detailed statistical information about the number of eigenvalues in an interval. Then we will discuss the connection between orthogonal polynomials and Riemann–Hilbert problems, the aim being to compute the behavior of $K_N(x,y)$ for $N \to \infty$.

Summary of the previous lecture

Recall that to compute the probability that there are no eigenvalues in an interval (a,b), we considered the characteristic function of the interval,

$$\chi(x) = \chi_{(a,b)}(x) = \left\{ \begin{array}{cc} 1 & \text{if } x \in (a,b) \\ 0 & \text{if } x \in \mathbb{R} \setminus (a,b) \end{array} \right\} \tag{1}$$

Then we found the following formula,

$$\text{Prob } \{ \text{ no eigs in } (a,b)\} = \mathbb{E}\left(\prod_{j=1}^{N} (1-\chi(\lambda_j)) \right). \tag{2}$$

which eventually led to the function

$$H(a,b,t) := \mathbb{E}\left(\prod_{j=1}^{N} (1-t\chi(\lambda_j)) \right), \tag{3}$$

and using a great deal of trickery, we were able to show that this has a representation in terms of a Fredholm determinant:

$$H(a,b,t) = \det\left(1 - t\mathcal{K}_N\right). \tag{4}$$

Expanding the product, we find

$$H(a,b,t) = 1 - t\mathbb{E}\left(\sum_{j=1}^{N} \chi(\lambda_j) \right) + t^2 \mathbb{E}\left(\sum_{1 \le i_1 < i_2 \le N} \chi(\lambda_{i_1})\chi(\lambda_{i_2}) \right) + \tag{5}$$
$$+ \sum_{j=3}^{N} (-t)^j \mathbb{E}\left(\sum_{1 \le i_1 < \cdots < i_j \le N} \prod_{k=1}^{j} \chi(\lambda_{i_k}) \right).$$

Our interest is in studying the behavior of this function as $N \to \infty$, and the above formula lends itself to such analysis: if the operator $\mathcal{K}_N$ converges in the Trace norm, then the function $H(a,b,t)$ also converges!

Largest eigenvalue

A special version of the probability that there are no eigenvalues in an interval is the case $(a,b) = (a,\infty)$ for in this case we have

$$\text{Prob } \{\lambda_{\max} < a\} = H(a,\infty,1). \tag{6}$$

number probabilities

It is also possible, using the same considerations to study the probability that there are n eigenvalues in the interval (a,b):

$$\text{Prob}\{\# \text{ eigs } \in (a,b) = n\} = \mathbb{E} \sum_{1 \le i_1 < \cdots < i_n \le N} \left(\prod_{\ell=1}^{n} \left(\chi\left(\lambda_{i_\ell}\right)\right) \prod_{\substack{\ell' = 1 \\ i_{\ell'} \notin \{i_\ell\}_{\ell=1}^{n}}}^{N} \left(1 - \chi\left(\lambda_{i_{\ell'}}\right)\right) \right) \tag{7}$$
$$= (-1)^n \frac{d^n}{dt^n} H(a,b,t)\bigg|_{t=1}.$$

On the other hand, we can also ask for higher statistics concerning the number of eigenvalues within the interval (a,b). Let us define the random variable $\#(a,b,M)$ to be the number of eigenvalues in (a,b):

$$\#\left(a,b,M\right)=\#\left\{\text{ eigs }\in(a,b)\right\}\ . \tag{8}$$

We have already observed that

$$\mathbb{E}\left(\#\left(a,b,M\right)\right)=-\frac{d}{dt}H(a,b,t)\bigg|_{t=0}\ . \tag{9}$$

Let us compute the variance of this random variable:

$$\mathrm{Var}\left(\#\left(a,b,M\right)\right)=\mathbb{E}\left(\left(\#\left(a,b,M\right)-\mathbb{E}\left(\#\left(a,b,M\right)\right)\right)^2\right)=\mathbb{E}\left[\left(\#\left(a,b,M\right)\right)^2\right]-\left(\mathbb{E}\#\left(a,b,M\right)\right)^2\ . \tag{10}$$

Now we can compute the first expectation by our usual trickery:

$$\mathbb{E}\left[\left(\#\left(a,b,M\right)\right)^2\right]=\int_{\mathbb{R}^N}\sum_{j,k=1}^{N}\chi(\lambda_j)\chi(\lambda_k)\frac{1}{N!}\det\left(K_N(\lambda_m,\lambda_n)\right)_{N\times N}d^N\lambda$$

$$=\int_{\mathbb{R}^N}\sum_{j=1}^{N}\chi(\lambda_j)^2\frac{1}{N!}\det\left(K_N(\lambda_m,\lambda_n)\right)_{N\times N}d^N\lambda+2\int_{\mathbb{R}^N}\sum_{1\le j<k\le N}\chi(\lambda_j)\chi(\lambda_k)\frac{1}{N!}\det\left(K_N(\lambda_m,\lambda_n)\right)_{N\times N}d^N\lambda$$

$$=\int_a^b K_N(\lambda,\lambda)d\lambda+\frac{2}{2!}\int_a^b\int_a^b\det\left(K_N(\lambda_m,\lambda_n)\right)_{2\times 2}d\lambda_1 d\lambda_2\ .$$

Combining this with (10), we find

$$\mathrm{Var}\left(\#\left(a,b,M\right)\right)=\int_a^b K_N(\lambda,\lambda)d\lambda+\frac{2}{2!}\int_a^b\int_a^b\det\left(K_N(\lambda_m,\lambda_n)\right)_{2\times 2}d\lambda_1 d\lambda_2-\int_a^b\int_a^b K_N(\lambda_1,\lambda_1)K_N(\lambda_2,\lambda_2)d\lambda_1 d\lambda_2\ . \tag{11}$$

So apparently the final claim from Lecture 14 was wrong. But here is a corrected formula:

$$\mathrm{Var}\left(\#\left(a,b,M\right)\right)=\frac{d^2}{dt^2}\log H(a,b,t)\bigg|_{t=0}=\frac{H(a,b,t)H''(a,b,t)-H'(a,b,t)^2}{H(a,b,t)^2}\bigg|_{t=0}=H''(a,b,0)-H'(a,b,0)^2 \tag{12}$$

Orthogonal Polynomials and Riemann–Hilbert problems

The following Riemann-Hilbert problem [1] is known to characterize the polynomials $p_j^{(N)}$ orthogonal with respect to $e^{-NV(x)}$.

Riemann-Hilbert Problem 1. *Find a 2×2 matrix $\mathbf{A}(z)=\mathbf{A}(z;n,N)$ with the properties:*

Analyticity. *$\mathbf{A}(z)$ is analytic for $z\in\mathbb{C}\setminus\mathbb{R}$, and takes continuous boundary values $\mathbf{A}_+(x)$, $\mathbf{A}_-(x)$ as z tends to x with $x\in\mathbb{R}$ and $z\in\mathbb{C}_+$, $z\in\mathbb{C}_-$.*

Jump Condition. *The boundary values are connected by the relation*

$$\mathbf{A}_+(x)=\mathbf{A}_-(x)\begin{pmatrix}1 & e^{-NV(x)}\\ 0 & 1\end{pmatrix}\ . \tag{13}$$

Normalization. *The matrix $\mathbf{A}(z)$ is normalized at $z=\infty$ as follows:*

$$\lim_{z\to\infty}\mathbf{A}(z)\begin{pmatrix}z^{-n} & 0\\ 0 & z^n\end{pmatrix}=\mathbb{I}\,. \tag{14}$$

It was discovered in [1] that Riemann-Hilbert Problem 1 characterizes polynomials orthogonal with respect to $d\nu(x) := e^{-NV(x)}\,dx$. The connection between these orthogonal polynomials and the solution of Riemann-Hilbert Problem 1 is the following:

$$\mathbf{A}(z) = \begin{pmatrix} \dfrac{1}{\kappa_{n,n}^{(N)}} p_n(z) & \dfrac{1}{2\pi i \kappa_{n,n}^{(N)}} \displaystyle\int_{\mathbb{R}} \frac{p_n(s) e^{-NV(s)}}{s-z}\,ds \\ -2\pi i \kappa_{n-1,n-1}^{(N)} p_{n-1}(z) & -\kappa_{n-1,n-1}^{(N)} \displaystyle\int_{\mathbb{R}} \frac{p_{n-1}(s) e^{-NV(s)}}{s-z}\,ds \end{pmatrix}. \tag{15}$$

This relationship provides a useful avenue for asymptotic analysis of the orthogonal polynomials in the limit $n \to \infty$; it is sufficient to carry out a rigorous asymptotic analysis of Riemann-Hilbert Problem 1.

In class we will present a proof that (15) represents the unique solution to Riemann–Hilbert 1.

References

[1] A. Fokas, A. Its, and A. V. Kitaev, "Discrete Painlevé equations and their appearance in quantum gravity", *Commun. Math. Phys.*, **142**, 313–344, 1991.